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TREATISE
ON
PLANE AND SPHERICAL TRIGONOMETRY:

WITH THEIR MOST USEFUL PRACTICAL APPLICATIONS.

BY JOHN BONNYCASTLE,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY,
WOOLWICH.

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INTRODUCTION.

At what period Trigonometry first began to be cultivated, as a branch of the mathematical sciences, is extremely uncertain, no records having been left by the ancients, which enable us to trace it to a higher age than that of Hipparchus, who flourished about a century and a half before Christ, and is reported by Theon, in his Commentary on Ptolemy's Almagest, to have written a work, in twelve books, on the chords of circular arcs, which, from the nature of the title, must evidently have been a treatise on Trigonometry.

But the earliest work on the subject, now extant, is the Spherics of Theodosius, a native of Tripoli in Bythinia, who, soon after the time above mentioned, collected the scattered principles of the science, which had been discovered by his predecessors, and formed them into a regular treatise, in three books, containing a variety of the most necessary and useful propositions relating to the sphere, arranged and demonstrated with great perspicuity and elegance, after the manner of Euclid's Elements(*a*).

The next of the Greek writers, after Theodosius,

(*a*) This work of Theodosius, which came to us through the medium of an Arabic version, has been published both in Greek and Latin by several writers; but the Latin edition of Dr. Barrow, 8vo. London, 1675, and that of Hunt, 8vo. Oxford, 1709, are who has treated professedly on this subject, is Menelaus, an astronomer and mathematician of considerable eminence, who lived about the middle of the first century after Christ, and of whom we have three books on Spherical Triangles, containing, besides the first principles of the science, a number of propositions of a more difficult kind, which at that time were but little known; but the six books which he is said to have written on the subtenses, or chords, of circular arcs, being probably a treatise on the ancient method of constructing trigonometrical tables, has not been transmitted to our times (*b*).

This loss, however, has been, in some measure, repaired by Ptolemy, who in the first book of his Almagest, published about the beginning of the second century after Christ, has given us a table of arcs and their chords to every half degree of the semicircle; in the forming of which it is observable, that he divides the radius, and the arc whose chord is equal to the radius, each into sixty equal parts, and then estimates all other arcs by sixtieths of that arc,

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reckoned the best. The third book, which is the most difficult, has been commented upon and elucidated by Pappus, in his Mathematical Collections.

(*b*) A translation of the Spherics of Menelaus had been undertaken by Regiomontanus, but was first published by Maurolycus in Latin (Messanæ, 1558, fol.), together with the Spherics of Theodosius and his own. Halley also prepared a new edition of this work, corrected from a Hebrew manuscript, which was published in 8vo. 1758, without the preface which he had projected for it, by Costard, the author of a History of Astronomy.

and the chords by sixtieths of that chord, or of the radius; being probably the method used by Hipparchus and other ancient writers on this subject. He has here also proved, for the first time that we know of, that the rectangle of the two diagonals of any quadrilateral inscribed in a circle, is equal to the sum of the rectangles of its opposite sides (*c*).

After the time of Ptolemy and his commentator Theon, little more is known on this subject till about the close of the eighth century after Christ, when the ancient method of computing by the chords was changed for that of the sines, which were first introduced into the science by the Arabians; to whom we are also indebted for the several axioms and theorems which are at present considered as the foundation of our modern Trigonometry, as well as for some other propositions which such an alteration in the system naturally required.

The Arabians, however, though they had been long acquainted with the Indian, or decimal scale of arithmetic, now used, do not appear to have deviated from the Greeks in the sexagesimal division of the

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(*c*) Claudius Ptolemeus was born at Ptolemais in Egypt, and taught astronomy at Alexandria, where he died in the year of Christ 147, being the 78th year of his age. His Almagest, like most of the celebrated works of antiquity, has had many editors and commentators; but a good translation, both of this work and the Commentary of Theon, is still much wanted; the only tolerably complete Latin edition (published at Basil, 1551,) which we now possess, being that of George of Trebizonde, which was so severely and justly criticised by Regiomontanus.

radius, which continued in use till about the middle of the 15th century, when an alteration was first made by Purbach, a native of a small place of that name between Austria and Bavaria; who constructed a table of sines to a division of the radius into 600000 equal parts, and computed them for every ten minutes, or sixth part of a degree, in parts of this radius, by the decimal notation.

This project of Purbach was also still further prosecuted by his disciple and friend John Muller, commonly called Regiomontanus, of the little town of Mons Regius, or Konigsberg, in Franconia, who first began his mathematical career by extending and improving the tables of his master; but, afterwards, disliking that plan, as evidently imperfect, he computed a table of sines, for every minute of the quadrant, to the radius 1000000. He also introduced the tangents into this science, and enriched it with many theorems and precepts, which, except for the use of logarithms, renders the trigonometry of this author but little inferior to that of our times(*d*).

Soon after the period here mentioned, several other mathematicians also contributed to the advancement of this science, either by some useful alterations in

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(*d*) The Treatise of Regiomontanus, on Plane and Spherical Trigonometry, in five books, was written about the year 1464, and printed in folio at Nuremberg, 1533. In the fifth book, some of the problems relating to plane triangles are resolved by means of Algebra, a proof that this science was known in Europe before the treatise of Lucas de Burgo appeared.

the form of the tables, or by others improvements; among whom may be reckoned John Werner of Nuremburg, and Nicholas Copernicus of Thorn, in Prussia, the celebrated restorer of the true system of the world, who wrote a brief treatise on plane and spherical Trigonometry, with a description and construction of the canon of chords, nearly in the manner of Ptolemy; which tract, together with a table of sines and their differences, for every 10 minutes of the quadrant, to radius 100000, is inserted in the first book of his *Revolutiones orbium cœlestium*, published in folio, at Nuremburg, 1543.

To these cultivators and improvers of the science, we may likewise add Erasmus Rheinold, professor of mathematics in the academy of Wurtemburg, who published his *Canon fœcundus*, or Table of tangents, in 1553; and Maurolycus, abbot of Messina, in Sicily, one of the most able geometers of the sixteenth century, to whom we are indebted for the *Tabula benefica*, or Canon of secants, which came out about the same time.

But the most complete work on the subject, which had hitherto appeared, was a treatise, in two parts, by Vieta, printed in folio at Paris, 1579, during the author's lifetime, in the first part of which, entitled *Canon mathematicus seu ad triangula, cum appendicibus*, he has given a table of sines, tangents and secants for every minute of the quadrant, to radius 100000, with their differences; and towards the end of the quadrant, the tangents and secants are extended to

8 or 9 places of figures. They are also arranged like our tables at present, increasing from the left-hand side to 45°, and then returning backwards, from the right hand, to 90°; so that each number and its complement stand on the same line.

The second part of the volume, which is entitled *Universalium inspectionem ad canonem mathematicum liber singularis*, contains, besides a regular account of the construction of the tables, a compendious treatise on plane and spherical Trigonometry, with their application to a variety of curious subjects in geometry, mensuration, and other branches of mathematics; as also a number of particulars relating to the quadrature of the circle, the duplication of the cube, and similar problems; which are all treated of in a manner worthy the genius of the author (*e*).

Beside the performance above mentioned, there are, likewise, several other smaller tracts on trigonometrical subjects in the general collection of Vieta's works, published at Leyden in 1646, by Schooten; among which are the curious theorems, here first given by our author, relating to angular sections, or the multiples and submultiples of arcs; as also general formulæ for the chords, and consequently sines, of the sums and differences of arcs, and such as are in arithmetical progression; which have since been so extensively and usefully applied, both in this

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(*e*) This curious performance, which was published separately from the other works of Vieta, and without his name, is extremely scarce, few copies of it having ever reached this country.

science, and in some of the higher branches of the modern analysis (f).

The next writer on this subject, whose labours deserve particular notice, is George Joachim Rheticus, a pupil of Copernicus, and professor of mathematics in the university of Wittemburg, who formed the design of computing the trigonometrical canon, for every ten seconds of the quadrant, to 15 places of decimals; and though he was prevented from executing the whole of this laborious enterprise, he nevertheless accomplished that part of it relating to the sines and cosines; all of which he calculated according to his original plan, besides those of every single second of the first and last degrees; but was deterred from publishing the work, on account of the expense attending the impression.

Soon after his death, however, which happened in 1576, Valentine Otho, one of his disciples and friends, engaged, according to the dying request of Rheticus, to finish this great undertaking; and notwithstanding a variety of difficulties and obstacles, which retarded his labours, he at length gave it to the public, under the title of *Opus Palatinum de Triangulis* (Heidelbergæ, folio, 1596): in which work, we have, for the first time, an entire table of sines, tangents and

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(f) The demonstrations of most of the trigonometrical theorems in this work, relating to angular sections, were supplied by Alexander Anderson, at that time professor of mathematics at Paris, but a native of Aberdeen in Scotland.

secants, for every ten seconds of the quadrant, to ten places of decimals, with their differences; the five last figures of the former Table being here omitted.

But as this performance, though highly valuable in other respects, was afterwards found to contain a number of errors, particularly in the cotangents and cosecants, which the sines that Otho had employed were not sufficiently extensive to prevent, Bartholomew Pitiscus, an able mathematician of that time, undertook the revision of it. And having procured, with some difficulty, the original manuscript of Rheticus, he added to it an auxiliary table of sines of small arcs, to 21 places of decimals, for the purpose of supplying the defect above mentioned, and published the work, with the augmentations he had made to it, under the name of *Thesaurus Mathematicus*, &c. (Francfort, folio, 1613.)

This being done, he now recalculated, by means of the materials he had thus prepared, the cotangents and cosecants of the Opus Palatinum of Otho, to the end of the first six degrees of the quadrant; and as this rendered the work sufficiently exact for all astronomical purposes, even to fractions of seconds, he published the corrections in separate sheets, in 86 pages in folio, for the purpose of replacing those of the former impression. But the original work having been partly sold off, and its purchasers neglecting to procure the new sheets, these corrected copies are become so extremely rare, that few of them are to be found, either in the possession of individuals or in the public libraries (*g*).

About the close of the 16th century, several other persons also wrote on the subject of Trigonometry, and the construction of the triangular canon; among whom may be reckoned Philip Lansberg, a native of Zealand in Holland, who in 1591 published his *Geometria Triangulorum*, in four books, with the usual tables; being the first compendium of this kind in which the tangents and secants are continued to the end of the last degree of the quadrant, to 7 places of decimals.

The Trigonometry of Pitiscus, also, which was published at Francfort in 1599, is a very complete work, having been long considered, both with respect to the correctness of the tables, and its numerous practical applications, as the most com-

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(*g*) For a more detailed account of these valuable works, see a paper by Prony, in the Mémoires de l'Institut, vol. v., where he says that he knows of only two of the corrected copies of the Opus Palatinum that are now to be found; one being in the library of the Council of State at Paris, and the other that purchased by himself of the bookseller Duprat; both of which can be easily distinguished from the old work, by the difference of the colour of the paper and type, in the sheets that are changed.

The title of the corrected copies is as follows:

Georgii Joachimi Rhetici Magnus Canon Doctrinæ triangulorum ad decades secundorum scrupulorum, et ad partes 10000000000.

Recens emendatus à Bartholomæo Pitisco silesio. Addita est brevis commonefactio de fabricâ et usu hujus canonis, &c.

Canon hic unà cum brevi commonefactione de ejus fabricâ et usu, etiam separatìm ab opere palatino venditur. In Bibliopoleio harnischiano.

modious and useful treatise on the subject then extant.

To these writers may likewise be added Christopher Clavius, a German Jesuit, who, in the first volume of his works, printed at Mentz, 1612, in 5 vols. fol., has given us an ample and circumstantial treatise on Trigonometry, with tables of sines, tangents and seconds, for every minute of the quadrant, to 7 places of decimals, and in a form continued forwards to the end of the quadrant. The sines have also their differences set down to every second, and the construction of the tables is clearly explained, according to the methods of Ptolemy, Purbach, and Regiomontanus.

About the year 1600, Ludoph van Ceulen, a Dutch mathematician, of considerable talents, also published his well-known treatise *De Circulo et adscriptis*, in which he treats of the properties of lines drawn in and about a circle, and especially of chords, or subtenses, with the construction of the canon of sines. He here, likewise, determined the ratio of the diameter of a circle to its circumference, to 36 places of figures; showing that if the diameter be 1, the circumference will be 3.14159 26535 89793 23846 26433 83279 50288 &c.; which ratio, in imitation of the example of Archimedes, is said to have been engraved, by his order, on his tombstone in the churchyard at Leyden.

This curious tract, with some other of Ceulen's dissertations on similar subjects, was translated into Latin, and published at Leyden, in 1619, by Willebrord Snell; who has himself given, in his *Doctrinæ triangulorum canonicæ*, the construction of the sines, tangents and secants, together with a very useful synopsis of the calculation of triangles, both plane and spherical.

Francis van Schooten also published, at Amsterdam, in 1627, a table of sines, tangents and secants, in a small neat form, for every minute of the quadrant, to 7 places of figures, which has a great character for accuracy, being declared, by its author, to be without a single error; though this must not be understood of the last figure of the numbers, which is sometimes erroneous in excess, and sometimes in defect, by not being always set down to the nearest unit.

These are the principal writers on Trigonometry, and the tables of sines, tangents and secants, before the change that was made in the subject by the introduction of the logarithmic calculus, which first began to be employed in this science about the commencement of the 17th century, by its celebrated inventor Baron Napier, of Merchiston, in Scotland; who, in the year 1614, published his work entitled *Mirifici logarithmorum canonis descriptio* (*h*), which

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(*h*) The principles of logarithms, and the method of computing the tables, are not given by Napier in this performance, but were afterwards published by his son, Robert Napier, who in the year 1619 gave a new edition of his father's work, together with the *Logarithmorum canonis constructio*, and other pieces; in which performance it may be observed, that the geometrical method, used by Napier in computing his logarithms, is similar to that which was afterwards employed by Newton in the generation of magnitudes, in his doctrine of fluxions.

contains the logarithms of numbers, and the logarithmic sines, tangents and seconds, for every minute of the quadrant, together with the description and use of the tables.

But as the text, or descriptive part of this work, was in the Latin language, it was soon afterwards translated into English by Mr. Edward Wright, the inventor of the principles of what has been usually, though erroneously, called Mercator's Sailing; who having finished his manuscript, sent it to Edinburgh to be revised and improved by the author; but on his dying a short time after he had received it back, it was published, with a preface by Briggs, in the year 1616, by his son Samuel Wright, together with the tables, but each number to one figure less than in the original.

Soon after this, several other logarithmic tables, of a kind nearly similar to those of Napier, were published by Speidell, Ursinus, and Kepler; but being in general very compendious, and formed upon principles which have since been found incommodious in practice, they are now chiefly curious on account of the ideas and artifices displayed by their authors in their different modes of computing them; in which respect the performance of Kepler, though frequently abstruse and obscure, is particularly deserving of notice, both from the originality of his plan, and the able manner in which it is developed (*i*).

(*i*) The treatise of Kepler here mentioned, is entitled *Chilias logarithmorum ad totidem numeros rotundos, præmissa demonstratione legitima ortus logarithmorum eorumque usus*, &c. (Marpurg, 1624.)

The person, however, to whom we are chiefly indebted for the new and more advantageous form which this admirable mode of computation has since assumed, is Mr. Henry Briggs, at that time professor of geometry in Gresham College, London, and afterwards Savilian professor at Oxford; who, besides his eminent talents as a mathematician, has the merit of having first proposed, both to the public in his lectures, and to the illustrious inventor of the doctrine himself, that happy improvement in the system of these numbers, which consists in making the radix of the system 10, instead of 2.71828182845, &c. as was done by Napier; or, which is the same thing, by changing them from what are usually called *hyperbolic, or Napierian logarithms*, to the present *common* or *tabular logarithms*.

It may here likewise be further remarked, that Briggs, in addition to his other attainments, was also a most indefatigable calculator, having laboured from the beginning, with great zeal and diligence, at the computation of logarithms of this kind, of which he was the inventor and promoter. And, as the first fruits of his industry, he produced, in 1624, his *Arithmetica Logarithmica*, a stupendous work for

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To which perfor ance, the year following, he added a supplement, containing the logarithms of integer numbers, and of such of the natural sines as nearly coincide with them.

It may also be observed, that the work of Ursinus above mentioned, entitled *Trigonometria* (Cologne, 1624), is not unworthy of attention, as containing a table of natural sines and their logarithms, of the Napierian form, to every 10 seconds of the quadrant, which the author had been at great pains in computing.

so short a time; which contains the logarithms of all numbers from 1 to 20000, and from 90000 to 100000 to 15 places of figures, besides the index.

The table, however, being imperfect, the remaining logarithms were soon afterwards supplied by Adrian Vlacq, of Gouda, in Holland, who completed the 70 intermediate chiliads, and republished the *Arithmetica Logarithmica* at that place, with these additional numbers, in 1627 and 1628; in which state it contains the logarithms of all numbers, from 1 to 100000, to 10 places of decimals, together with a table of logarithmic sines, tangents and secants, to the same extent, for every minute of the quadrant.

To this we may also add, that beside the work above mentioned, Briggs lived to complete a table of logarithmic sines and tangents for the hundredth part of every degree, to 14 places of decimals, together with a table of natural sines for the same parts to 15 places, and the tangents and secants of the same to 10 places, with the construction of the whole; which work was likewise printed at Gouda, by Vlacq, in 1633; and on his death, a preface to it was supplied by Mr. Henry Gellibrand, at that time professor of astronomy in Gresham College, who also added to it the application of logarithms to plane and spherical trigonometry, and published it the same year, under the title of *Trigonometria Britannica.*

These two performances of Briggs also contain, besides the extensive tables above mentioned, and the method of constructing them, a variety of other matters of great utility and importance in this science; among the most remarkable of which may be mentioned, the method of interpolating logarithms by their differences, as afterwards treated of by Cotes, in his *Canonotechnia*, and the proof he has given, in his chapter on angular sections, of the curious property, that the sines of equidifferent arcs, with their 2d, 4th, 6th, &c. differences, and the cosines of the mean arcs, with their 1st, 3d, 5th, &c. differences, are in geometrical progression (*k*).

In the same year, likewise, and during the time that he was superintending the printing of the *Trigonometria Britannica* of Briggs, Vlacq published, at Gouda, his own great work, entitled *Trigonometria Artificialis*, which contains the logarithmic sines and tangents of every 10 seconds of the quadrant, to 10 places of figures, besides the index; and the logarithms of the first 20000 numbers, to the same number of places, with the differences of each; the whole being preceded by an ample description of the tables, and the application of them to some of the principal problems in plane and spherical Trigonometry (*l*).

(*k*) Besides what relates more immediately to trigonometrical subjects, Briggs has shown, in his *Trigonometria Britannica*, the method of generating the coefficients of the terms of any integral power of a binomial, successively from each other, independently of any other power; a property which Newton afterwards exhibited in the form of a general theorem, algebraically expressed, and serving for all kinds of powers or roots, whether integral or fractional.

(*l*) A new edition of the *Trigonometria Artificialis* of Vlacq, which has always been considered as a work of great use to Astronomers, has been lately published at Leipzig, by Vega, under the title of *Thesaurus Mathematicus*.

Several smaller tables of these logarithms were also published about the same time by Gunter, Wingate, Roe, and others; the two latter of whom considerably improved their form and disposition. Gunter is likewise deserving of notice, from his having first applied the logarithms of numbers, together with those of the sines and tangents, to a ruler, in the form of a two-foot scale, that still goes by his name: by which proportions in trigonometry, navigation, and other subjects, may be performed, to a degree of accuracy sufficient for many practical purposes, by the mere application of a pair of compasses; being a method founded on the well-known property, that the logarithms of the terms of equal ratios are equidifferent (*m*).

But the common logarithmic canon was first reduced to its most convenient form by John Newton, in his *Trigonometria Britannica*, printed at London in 1658, which work contains the logarithms of the first 100000 numbers, to 8 places of decimals, besides the index, arranged in the same manner as they are in our best tables at present; as also the logarithmic sines and tangents to the same extent, for every 100dth part of a degree, with their differences, and for the 1000dth part in the first three degrees, according to the decimal division of Briggs.

(*m*) Gunter, who was professor of astronomy at Gresham College at the time that Briggs was professor of geometry there, is also said to have first introduced the use of arithmetical complements into logarithmic computations, and to have been the inventor, or at least to have first started the idea, of the logarithmic curve.

The greater part of these tables, however, have since been, in some measure, superseded by those of a later date; among the most accurate and convenient of which, for common use, may be reckoned the edition of Vlacq's small volume of tables, printed at Lyons in 1670, and another work of this kind, printed at the same place, in 1760; but more particularly by the edition of Sherwin's Mathematical Tables, in 8vo. 1742, as revised by Gardiner; also Hutton's Mathematical Tables, in 8vo., first printed in 1785; the Tables of Vega, 2 vols. 8vo., printed at Leipzig in 1797; and the 1st edition of the Tables Portatives de Logarithmes of Callet, in small 8vo., printed at Paris 1783 (*n*); all of which are adapted to the sexagesimal division of the circle, used by Vlacq and most of the later compilers.

Besides these, several other tables, of a different kind, have been lately published by the French; in which the quadrant is divided, according to their new system of measures, into 100 degrees, the degree into 100 minutes, and the minute into 100 seconds; the principal of which are the 2d edition of the Tables Portatives of Callet, beautifully printed in stereotype, at Paris, by Didot, 8vo. 1795, with great additions and improvements; the Trigonometrical Tables

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(*n*) This neat portable work, which is now become extremely scarce, contains all the tables in Gardiner's 4to vol. hereafter mentioned, with several additions and improvements; and is, by far, the most useful and convenient performance of the kind that has yet been offered to the public.

of Borda, in 4to. an. ix, revised and enriched with various new precepts and formulæ by Delambre; and the tables lately published at Berlin, by Hobert and Ideler, which are also adapted to the decimal division of the circle, and are highly praised for their accuracy by the French computers.

Among the various tables, however, of the sexagenary kind, none have been more esteemed for their usefulness and accuracy than those of Gardiner, printed in 4to. at London, in 1742; which contain the logarithms of all numbers from 1 to 102100, and the logarithmic sines and tangents for every ten seconds of the quadrant, to 7 places of decimals, with several other necessary tables; a new edition of which work was also printed at Avignon, in France, in 1770, under the care of Pezenas, who added to it the sines and tangents of every single second, for the first 4 degrees, and a small table of hyperbolic logarithms, taken from Simpson's Fluxions.

But of all the trigonometrical tables hitherto published, the most extensive and best adapted for obtaining accurate results, in many delicate astronomical and geodetical observations, are those of Taylor, printed in large 4to. at London, 1792; which contain the logarithms of the first common numbers from 1 to 1260, to eight places of decimals; the logarithms of all numbers from 1 to 101000, to 7 places; and the logarithmic sines and tangents of every second in the quadrant, to 7 places; as also a preface, and various precepts for the explanation and use of the tables, which, from the author's dying before the last sheet of his work was printed off, were supplied by Dr. Maskelyne, the astronomer royal.

It may here likewise be observed, that besides the common tables hitherto mentioned, which contain the logarithms of numbers in their usual order, others, of a different kind, have been constructed, for the more readily finding the number corresponding to any given logarithm; of which the principal one, of any considerable extent, is the Antilogarithmic Canon of Dodson, published at London, in 1742; which contains the numbers corresponding to every logarithm, from 1 to 100000, to eleven places of figures, with their differences and proportional parts; and, though little used at present, is a performance of great labour and merit (*o*).

In consequence also of the decimal division of the circle, now generally used by the French mathematicians, a number of persons have been employed, for several years past, at the Bureau de Cadastre, at Paris, under the direction of Prony, in computing new trigonometrical tables of this kind, to a far greater extent than any that have hitherto been devised. But notwithstanding the work appears to

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(*o*) Dr. Wallis informs us, in the 2d vol. of his mathematical works, that an antilogarithmic canon was begun by Harriot, the algebraist (who died in 1621), and finished by Warner, the editor of his works, about the year 1640; but which was lost for want of encouragement to print it.

have been nearly completed a considerable time since, it has not yet been offered to the public: which is much to be regretted; as these Tables, though too bulky and high priced to be brought into common use, might be very advantageously consulted, in many points of delicate calculation, which are otherwise not easily determinable. Our common tables could likewise be corrected from them, or new ones published under an abridged form; it is therefore to be hoped, that this great monument of calculation will soon make its appearance, under the auspices of a government which declares itself to be *l'ami des Arts et des Sciences* (*p*).

To this brief account of the works of some of the most celebrated writers on the subject here treated of, and the tables which, at different times, have been composed for facilitating its practical operations, it may be proper to subjoin a slight sketch of the improvements which it has undergone in passing through the hands of the later analysts, who, by means of a more commodious algorithm, and the resources of a ready and comprehensive calculus, unknown to their predecessors, have greatly enlarged the boundaries of the science, and simplified its rules and processes.

These advantages, and the consequent discoveries which attended them, have chiefly arisen from the

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(*p*) For a detailed account of the contents of this great work, and the manner in which it was computed, see the Report of Delambre, Mémoires de l'Institut, vol. v.

new views of the subject that had been opened to mathematicians, by the theorems, first given by Vieta, for the chords of the sums, differences, and multiples of arcs and their supplements; which though left without demonstration, and, in the latter case, probably formed by induction from the law of the terms and their coefficients, have, nevertheless, been the germ of most of the numerous and elegant formulæ which have since enriched this branch of the subject.

We are also, in this respect, no less indebted to Napier, not only for his admirable discovery of logarithms, but for the new and excellent analogies which he introduced into that part of the science relating to the solution of spherical triangles, which still go by his name; as likewise for his other well known rules, called the Five Circular Parts (q): which, though too artificial and restricted in their application, to be employed by skilful computists, in the present advanced state of the science, are sufficient proofs of the dexterity and address with which he investigated every branch of a subject so intimately connected with the invention that has gained him such just celebrity.

(q) It is an exaggerated account of the excellence of the two rules here mentioned, to assert, as some writers have done, that the bare recollection of them, alone, is sufficient to enable a person to carry the whole of spherical Trigonometry in his mind; for, besides their not being applicable to the two cases in which the 3 sides, or the 3 $\angle^s$, of a $\triangle$ are given, a number of other particulars, not easily remembered, are equally necessary to be known, respecting the falling of the perpendiculars within or without the $\triangle^s$, and the affections of the sides and angles, before a complete solution of them can be obtained.

The works of Briggs, already mentioned, also greatly contributed to the advancement of this branch of the science, both by the assistance which they afforded to the practical calculator, in many intricate and abstruse computations, and by the numerous improvements and discoveries of a higher kind, with which they abound. The method, in particular, which he appears to have first used in raising logarithms from their differences, and his skilful application of analytical principles to several other subjects of difficult investigation, entitle him to rank with the first mathematicians of the age in which he lived.

The logarithmic curve, likewise, and those of a similar kind, which first began to be introduced about this time, greatly facilitated the conception of these numbers, by exhibiting some of their most remarkable properties in a more perspicuous way than could be done by the abstract methods of investigation employed by Napier and others. And though the doctrine itself has no necessary connexion with these or any other geometrical figures, it was from this source that the new and advantageous mode of expressing logarithms by series was first derived.

This happy improvement, which was introduced into the science about the year 1668, by Mercator and James Gregory, who were led to the discovery of some of the most simple forms of these series by contemplating the nature of the hyperbola, was soon afterwards extended to the trigonometrical part of the subject, or the arithmetic of sines, as it is now frequently called; which Newton, Leibnitz, the Bernoullis, and others, enriched with similar formulæ; and by this means assimilated the principles of logarithms and trigonometry with those of the new calculi, of which they were the inventors and improvers.

The exponential formulæ, also, for the sines and cosines of arcs, which were first given by Demoivre, have greatly contributed to the progress of the analytical branch of this subject, by abridging its operations, and shortening the labour of investigation; and though some writers, who appear to be unacquainted with the way in which imaginary symbols arise, have represented all expressions of this kind as nugatory and absurd, yet their commodious form, and the ease and certainty with which they can be applied in many intricate inquiries, will always cause them to be regarded by the skilful analyst as an important acquisition to the science (*r*).

Many other improvements, of more or less importance, have since been made, both in the practical and theoretical branches of this subject, by later writers; but of these, none have proved of such

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(*r*) Lagrange, one of the most profound analysts of the present age, regards the exponential formulæ of Demoivre, here alluded to, for expressing the sines and cosines by means of certain imaginary functions of the arcs, as one of the greatest analytical discoveries of the 17th century.

general advantage to the science as the substitution of the analytical mode of notation in the place of the geometrical; which useful change was first introduced by Euler; who, besides this simplification of the former methods, has developed and extended, in his numerous works, almost every part of the trigonometrical analysis; which, under his masterly hand, assumed the form of a new science.

With respect to the projection of the sphere, which is also a branch of science intimately connected with this subject, little more is known of the early part of its history than what can be collected from the writings of Ptolemy, who in his treatise on the planisphere, as well as in his geography, has left us a number of propositions relating to the stereographic method, as it is now generally called, of representing the surface of the sphere upon the plane of one of its great circles, and its application to the construction of maps and charts (*s*).

It is evident, however, that Ptolemy, whose work above mentioned is the oldest of the kind that we now possess, was not the author of this mode of projection, as he has commonly been thought to be; since it appears from a letter, still extant, addressed

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(*s*) It may here be observed, that the term Stereographic, which denotes solidity, or a projection of three dimensions, is an improper appellation; and though the word is of Greek origin, its application to this subject is modern, having been first proposed and employed by Aguilon, in his Optics, printed at Anvers, 1613; before which time it had the name of Planisphere.

by Synesius, a disciple of the celebrated Hypatia, and afterwards bishop of Ptolemais in Libya, to Pæonius, on his sending him a silver astrolabe, that the method of exhibiting the surface of the sphere in this manner, upon a plane, was known at an earlier period, and had been particularly employed, for geographical and other purposes, by Hipparchus, to whom he expressly ascribes the invention.

But though the foundation of this doctrine was established by the ancients, they appear to have been unacquainted with some of its most important principles; as it was not known to Ptolemy that all the circles of the sphere, excepting those whose planes pass through the eye, are, in the stereographic method, projected into circles; or that any two projected great circles make the same angle, by their intersection on the plane of projection, which the original circles make on the sphere.

The first of these remarkable properties was, in fact, for a long period within the reach of mathematicians, without their availing themselves of it. For Apollonius having demonstrated, in his Conics, that the subcontrary section of a cone is a circle, the only thing that remained to be done was to prove that the plane of projection, in this method, forms a subcontrary section in every cone, the vertex of which is the eye, and the base a circle of the sphere; but, easy as this step appears, it was not achieved till fifteen hundred years after the time of Hipparchus.

It is uncertain, indeed, by whom, or at what time, these two useful propositions were first introduced into this branch of the subject, as the latter is not to be found in the large treatise of Clavius on the astrolabe; and though the former is distinctly mentioned by Jordanus in his Planisphere, printed at Bâle in 1536, the first clear and rigorous demonstration of it, by means of the subcontrary section, was given by Commandin, in his Commentary on the Planisphere of Ptolemy, published at Venice, 1558. The same obscurity also attends whatever relates to the origin and progress of the orthographic and gnomonical projections, of which no account is to be found in any of our mathematical histories, though the theory and practice of these methods have been amply treated of by several writers, who have neglected, with singular indifference, all inquiries concerning the authors and improvers of these useful inventions.

ADVERTISEMENT

TO

THE SECOND EDITION.

Before the time of the first publication of the present performance, no work of a similar kind had appeared, in this country, which could be considered as partaking, in any degree, of the great discoveries and improvements which had been made, in the subject here treated of, within the last fifty years.

The author was therefore induced to lay before the public an additional treatise on this useful and interesting branch of knowledge, that should be more conformable, both in matter and method, to the present state of the science; and from the favourable reception it has met with, he flatters himself, that the undertaking has not been regarded as either useless or unnecessary.

A careful revision, however, of the several parts of the work, appeared to the author to be still wanting; and he has accordingly made such alterations and improvements in the present Edition of it, without materially departing from his original plan, as he trusts will render it still more deserving the public favour.

Among other things, of this kind, that might be mentioned, nearly all the practical questions before given, many of which had passed, in the same state, from one work to another, for more than a century past, are here reproposed with new data; and in every case, where it was judged necessary, both the examples and the rules upon which they depend, have been carefully attended to, and reduced to their most commodious and approved forms.

The analytical part of the work, also, where the author first introduced to the notice of the English student, the many elegant theorems and formulæ, with which Euler, Delambre, Legendre, and other eminent mathematicians have enriched the science, is here considerably improved; and, though comprised within a short compass, will, it is conceived, be found to contain all the most useful and essential principles of this extensive and important branch of analysis.

J. BONNYCASTLE.

Royal Military Academy,
Oct. 17th, 1812.

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BONNYCASTLE'S

TRIGONOMETRY.

PLANE TRIGONOMETRY.

Plane Trigonometry is the science which treats of the analogies of plane triangles, and of the methods of determining their sides and angles.

It also comprehends whatever relates to the properties and relations of certain right lines drawn in and about a circle, called sines, tangents, &c.

The sides of plane triangles are estimated in feet, yards, chains, &c; or by abstract numbers, according to the purpose intended.

A right lined angle is measured by an arc of a circle contained between its two legs, and having the angular point for its centre (*a*).

Every circle is supposed to be divided into 360 equal parts, called degrees; each degree into 60 equal parts, called minutes; each minute into 60 equal parts, called seconds, &c.

So that a semicircle, or half the circumference, con-

(*a*) It is proved by Euclid, and in B. vi. prop. 24 of my Geometry, that angles at the centres of equal circles (and consequently at the centre of the same circle) are proportional to the arcs on which they stand; and in B. viii. prop. 6, that the circumferences of different circles, or any similar parts of them, are as the radii of those circles; from which properties, the reason for assuming arcs of circles as the measures of angles, will be sufficiently evident.

tains 180 degrees; and a quadrant, or fourth part of the circumference, 90 degrees: also a sextant is an arc of 60 degrees, and an octant an arc of 45 degrees.

An angle is likewise said to be of as many degrees, minutes, seconds, &c, as there are divisions of this kind contained in the arc, or part of the circumference, by which it is measured (*b*).

Hence, a right angle, being measured by a quarter of the circumference, is 90 degrees; an obtuse angle is greater than 90 degrees; and an acute angle less than 90 degrees.

Degrees are marked at the top of the figure, or figures, by which the arc is denoted, by a small °, minutes by ′, seconds by ″, &c.

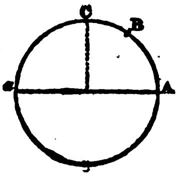
Thus, $39^\circ\ 28'$ is 39 degrees, 28 minutes; and $47^\circ\ 12'\ 19''$ is 47 degrees, 12 minutes, and 19 seconds.

The complement of an arc is the difference between that arc and a quadrant, or 90°; and the supplement of an arc is the difference between that arc

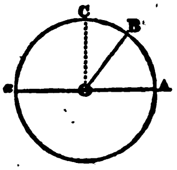
---

(*b*) From this method of estimation, it is plain that an angle may be of any magnitude less than 180 degrees; and that it is the same thing whether the circle which is described from the angular point be larger or smaller, as the arc contained between the two legs is always the same part of the whole circumference. It may also be remarked, that it is not the angles of triangles themselves that are employed in calculation, but certain lines called the sines, tangents, &c of these angles; as the latter can always be compared with the sides of the triangle, being quantities of the same kind, which is not the case with the angles, or arcs by which they are measured.

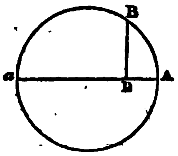
and a semicircle, or 180°. Thus, B C is the complement of A B; and B *a* is its supplement (*c*).



In like manner, the complement of an angle is the difference between that angle and a right angle, or 90°; and the supplement of an angle is the difference between that angle and two right angles, or 180°. Thus B O C is the complement of A O B, and B O *a* is its supplement.



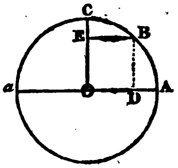
The sine of an arc is a right line drawn from one extremity of the arc perpendicular to the diameter which passes through the other extremity. Thus B D is the sine of A B, or of B *a*.



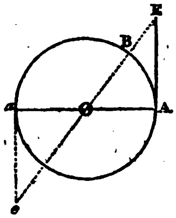
The cosine of an arc is the sine of the complement

(*c*) The complement is common to two arcs, or angles, which are the supplements of each other. Thus, B C is the complement of A B or of B *a*; but in most practical questions, it is usually understood to be what any acute angle, or an arc, wants of 90°.

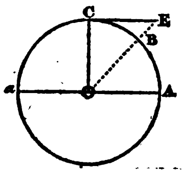
of that arc, or the part of the diameter which lies between the centre of the circle and the sine. Thus B E, or its equal O D, is the cosine of A B or of B *a*, or the sine of its complement B C.



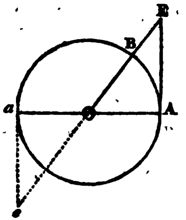
The tangent of an arc is a right line drawn perpendicular to the diameter, at one extremity of the arc, and terminated by a right line drawn from the centre through the other extremity. Thus A E is the tangent of A B, or of B *a*.



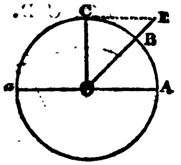
The cotangent of an arc is the tangent of the complement of that arc. Thus C E is the cotangent of A B, or of B *a*, or the tangent of its complement B C.



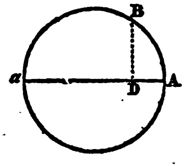
The secant of an arc is a right line drawn from the centre, through one extremity of the arc, and terminated by the tangent, or a line drawn perpendicular to the diameter at the other extremity. Thus O E is the secant of A B, or of B *a*.



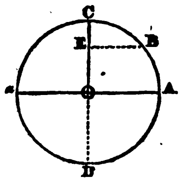
The cosecant of an arc is the secant of the complement of that arc. Thus O E is the cosecant of A B, or of B *a*, or the secant of its complement B C.



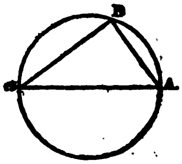
The versed sine of an arc is that part of the diameter which lies between the beginning of the arc and the sine. Thus A D is the versed sine of A B, and D *a* is the versed sine of its supplement B *a*.



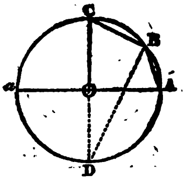
The coversed sign of an arc is the versed sine of the complement of that arc. Thus C E is the coversed sine of A B, or the versed sine of its complement B C.



The chord of an arc is a right line joining the two extremities of that arc. Thus A B is the chord of the arc A B, and B a is the chord of its supplement (d).



The cochord of an arc is the chord of the complement of that arc. Thus B C is the cochord of A B, or the chord of its complement B C; and B D is the chord of the supplement of C B.



The lines here described, belong equally to an angle as to the arc by which it is measured; and, except the chords and versed-sines, they are all common to two arcs or angles which are the supplements of each other.

So that if the sine, tangent, &c of any arc or angle above 90° be required, it is the same thing as

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(d) The sine or cosine of any arc or angle can never exceed the radius, and the secant and cosecant are never less than the radius; but the tangent and cotangent admit of all possible degrees of magnitude.

It may likewise be remarked that the chord of 60°, the sine and versed sine of 90°, and the tangent of 45° are all equal to the radius, whatever may be the magnitude of the circle; also the sine of 30°, and the cosine, or versed sine, of 60°, are half the radius, and the secant of 60° is double the radius.

to find the sine, tangent, &c of its supplement, or what it wants of 180°.

It may also be remarked, that these lines are called the natural sines, tangents, &c of the arcs or angles to which they belong; and the logarithms of the numbers by which they are represented, are the logarithmic sines, tangents, &c.

And as one or other of these make a part of every trigonometrical operation, they have been calculated to a given radius, for every degree, minute, &c of the quadrant, and ranged in tables for use.

Whence, by the help of such a table, the sine, tangent, &c of any arc or angle may be found by inspection; and, vice versa, the arc, or angle, to which any sine, tangent, &c belongs.

Upon these tables, also, and the doctrine of similar triangles, depends the solution of the several cases of plane trigonometry, which may be performed either by the natural or logarithmic sines, tangents, &c, as occasion requires.

But the logarithmic sines, tangents, &c, are those most commonly used; as the calculations, in this case, are all performed by adding and subtracting only, instead of multiplying and dividing, as is required by the natural sines, &c (*e*).

(*e*) The sine, tangent, &c of any arc or angle being of the same magnitude as the sine, tangent, &c of its supplement, it is plain that a table of these lines made for every degree, minute, &c of the quadrant, or 90°, will serve for the whole circle.

It is also to be observed, that, in every such table, the natural sines, tangents, &c are usually calculated to radius 1; but in order

In every plane triangle, three things must be given to find the rest; and of these three one at least must be a side, because the same angles are common to an infinite number of triangles.

These circumstances having been stated, it may be observed, that all the varieties that can possibly happen in the solution of plane triangles, are comprised in the three following cases: viz.

1. When two of the three given things are a side and its opposite angle.

2. When two sides and their included angle are given.

3. When the three sides are given.

Each of which cases may be resolved, either by geometrical construction, by arithmetical computation, or instrumentally.

In the first of these methods, the triangle is constructed by laying down the sides from a scale of equal parts, and the angles from a scale of chords, or a protractor, and then measuring the unknown

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that the logarithmic sines, tangents, &c. may be all positive, they are calculated to radius 10000000000, or 1 with 10 ciphers, the logarithm of which is 10, so that the latter are the logarithms of the former with 10 added to the index.

And as the natural sines, tangents, &c of any angles, or arcs of different circles, are proportional to the radii of those circles, their values may be readily found, or made to correspond to any radius whatever.

Such circumstances as relate to the state of the sine, tangent, &c with respect to their being positive or negative, will be noticed in another part of the work, as they do not interfere with the common practice.

parts by the same scale or instrument from which the others were taken.

In the second method, having stated the proportion, according to the proper rule, multiply the second and third terms together, and divide the product by the first, and the quotient will be the fourth term required, for the natural numbers. Or, in working by logarithms, add the logarithms of the second and third terms together, and from the sum take the logarithm of the first; then the number answering to the remainder will be the fourth term sought.

In the third method, or instrumentally, as suppose by the logarithmic lines on one side of the common two foot scales, extend the compasses from the first term to the second or third, as they may happen to be of the same kind; and that extent will reach from the other term to the fourth term required, taking both extents towards the same end of the scale.

The second of these methods, however, or that in which the operation is performed by logarithms, is the one generally employed; the other two being chiefly of use as checks on the calculations, or, in certain simple cases, where a near approximate value of the quantities to be determined is thought sufficient (*f*).

But it may here be remarked, that when one or more logarithms are to be subtracted, in any operation,

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(*f*) In working any question by logarithms, it is not always necessary to make an accurate construction of the figure, as the learner should accustom himself to such as are readily formed by the pen, and used only for the purpose of guiding him in the calculation.

it will be better to write down their *complements*, or what each of them wants of 10, instead of the logarithms themselves, and then add them all together, abating as many tens in the index of the sum as there are logarithms to be subtracted.

Thus, if the logarithm to be subtracted be 3.4932758, it is the same thing as to add its complement 6.5067242, and then abate 10 in the index of the sum; and if the logarithm be 9.0743260, it is the same as to add its complement 0.9256740, and abate 10 as before; observing that the readiest way of doing this, is to begin at the left hand and subtract each figure of the logarithm from 9, except the last significant figure on the right, which must be subtracted from 10.

If the index of the logarithm, whose complement is to be taken, be greater than 10, write down what it wants of 19, and take the rest of the figures from 9 as before; observing, after the addition is made, to abate 20 from the index of the sum. And if the logarithm of a decimal, which has always a negative index, is to be subtracted, add the index, considered as affirmative, to 9; and then find the complement of the rest of the figures as usual; and abate 10 in the index of the sum.

Thus, the complement of the logarithm 12.4907327 is 7.5092673; and the complement of the logarithm 2.5972648 is 11.4027352 (*g*).

(*g*) The truth of this method of computation, which arises from adding the complements of the logarithms instead of subtracting the logarithms themselves, may be rendered obvious as follows:

When the index of the logarithm (L) is under 10, or between

PROPERTIES OF PLANE TRIANGLES, REQUIRED IN THE PRACTICAL PART OF THE SCIENCE.

1. The sum of all the three angles of any plane triangle is equal to two right angles, or 180° (*h*).

2. The greater side of any plane triangle is opposite to the greater angle; and the less side to the less angle.

3. The sum of any two sides of a plane triangle is greater than the third side; and the difference of any two sides is less than the third side.

4. The triangle is equilateral, isosceles, or scalene, according as its three angles are all equal, or only two of them equal, or all three unequal.

5. The angles opposite to the two least sides of any

---

10 and 20, which is as far as any practical use of the tables extends, it is plain that $-\text{L} = (10 - \text{L}) - 10$; or $-\text{L} = (20 - \text{L}) - 20$, agreeably to the rule.

But if the index of the logarithm (L) to be subtracted be negative, let it be denoted by i, and the decimal part by d; then $-\text{L} = -(-i + d) = +i - d = (9 + i) \times (1 - d) - 10$, which is, also, the same as the rule.

The method of obtaining the complement, by beginning at the left hand, and subtracting each figure of the logarithm, except the last, from 9, is the same in effect as subtracting them from 10 in the usual way, and is merely employed upon this occasion as being more commodious.

(*h*) Since the sum of all the three angles of any plane triangle is 180°, if one of the acute angles of a right-angled triangle be subtracted from 90°, the remainder will be the other acute angle.

In like manner, if the sum of any two angles of a plane triangle be taken from 180°, it will leave the third angle; and if any one of the three angles be taken from 180°, it will leave the sum of the other two.

plane triangle are acute; and if there be an obtuse angle, it will be opposite to the greatest side.

6. A perpendicular drawn from the opposite angle of any plane triangle to the longest side, will fall within the triangle; and the greater segment will lie next the greater side, and the least segment next the least side (*i*).

7. In any isosceles plane triangle, a perpendicular, drawn from the vertex, will bisect both the base and the vertical angle.

8. In a right-angled plane triangle, the hypothenuse is equal to the square root of the sum of the squares of the other two sides; and either of the sides is equal to the square root of the difference of the squares of the hypothenuse and the other side.

Note, also, that if the half difference of any two quantities be added to their half sum, it will give the greater of those quantities; and, if it be subtracted from the half sum, it will give the less.

CASE I.

When two of the three given things are a side and its opposite angle, to find the rest.

(*i*) Besides the case here mentioned, which is sufficient for all practical purposes, it may be further observed, that if the angles at the base be both acute, the perpendicular will fall within the triangle; but, if one of them be obtuse, it will fall without the triangle, on the side of the obtuse angle. And in either of these cases, as in the former, the greatest segment will lie next the greatest side, and the least segment next the least side.

RULE.

The sides of any plane triangle are to each other as the sines of their opposite angles; and vice versa: —That is,

As any side is to the sine of its opposite angle, so is any other side to the sine of its opposite angle: or as the sine of any angle is to its opposite side, so is the sine of any other angle to its opposite side.

Hence, to find an angle, begin the proportion with a side opposite to a given angle; and to find a side, begin with an angle opposite to a given side.

Note. When two sides and an angle opposite to one of them are given, to find the rest, the question is sometimes ambiguous, or will admit of two different answers.

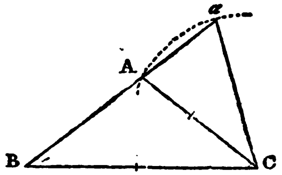
Thus, if the given angle be opposite to the least of the two given sides, the angle to be found, by the rule, may be either an acute angle or its supplement; but, if it be opposite to the greater side, the required angle will be acute (*k*).

(*k*) The ambiguity in this case does not arise, as has commonly been thought, from the required angle being found by means of its sine, which is the same both for an acute angle and its supplement, but solely from circumstances attending the construction of triangles from given data. For if the side C A, in the following figure, be proposed of such a length that it shall be less than C B, and greater than a perpendicular drawn from C to the opposite side, it must *necessarily* cut the indefinite line B A *a* in two points; while, in all other cases, it will either fall short of it, or touch it only in one point. Hence, when C A is proposed less than C B × sin B, the triangle is impossible: when it is greater than C B there can be

EXAMPLE I.

In the plane triangle A B C,

Given $\begin{cases} \text{A C } 248 \\ \text{B C } 354 \\ \angle\text{B } 41^\circ\ 36' \end{cases}$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ Yards. Required the other parts.



BY CONSTRUCTION.

1. Lay down the line B C = 354, from some convenient scale of equal parts; and make the ∠ B = 41° 36′ by a scale of chords, or other instrument.

2. With the centre C, and radius 248 (A C) taken from the same scale of equal parts, cross B A in A or *a*, and join C A or C *a*, and A B C, or *a* B C, will be the triangle required.

Then the ∠s C and A, measured by the scale of chords, and the side B A, or B *a*, by the scale of equal parts, will be found to be as follows, viz.

∠ C 29½°	∠ A 71¼°	A B 185½
or 67°	or 108¾°	or 343¾

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only one triangle; and if it fall between these limits, the triangle is ambiguous, or admits of two different answers.

Thus, in the above figure, where the least side A C is opposite to the given acute angle B, it is evident that either A B C or *a* B C may be the triangle sought. But when the given angle is right or obtuse, it will be opposite to the greatest side; and in this case there can be no ambiguity; for then neither of the other angles can be obtuse, and the geometrical construction will accordingly give only one triangle.

BY CALCULATION.

As side AC - - - 248 - - - - 2.3944517
7.6055483
Is to sine ∠ B - - 41° 36′ - - 9.8221198
So is side B C - - 354 - - - - 2.5490033
To sine ∠ A 71° 23′ or 108° 37′ 9.9766714
41° 36′ 41° 36′
Sum 112° 59′ or 150° 13′
Subtract 180° 0′ 180° 0′
Leaves 67° 1′ or 29° 47′ ∠ C

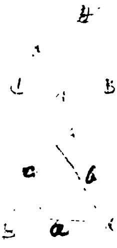
Then,
: Sine ∠ B 41° 36′ - - - - - - 9.8221198
0.1778802
: Side A C 248 - - - - - - - - 2.3944517
: : Sine ∠ C 29° 47′ - - - - - - 9.6962012
: Side A B 185.58 - - - - - - 2.2685321

Or,
: Sine ∠ B 41° 36′ - - - - - - 9.8221198
0.1778802
: Side A C 248 - - - - - - - - - 2.3944517
: : Sine ∠ C 67° 1′ - - - - - - - 9.9640797
: Side *a* B 343.88 - - - - - - 2.5364116

INSTRUMENTALLY.

In the first proportion, extend the compasses from 248 (A C) to 354 (B C) upon the line of numbers, and that extent will reach, upon the sines, from 41⅔° (∠ B) to 71½°, for the ∠ A.

In the 2d proportion, extend from 41⅔° (∠ B) to 29¾° or 67° upon the sines, and that extent will reach, upon the line of numbers, from 248 to 185½, or 343¾ for the side A B, or *a* B.



EXAMPLE II.

In the plane triangle A B C,

Given	Ans.
A B 237 c	A C 131.78
B C 197 a	∠ A 56° 13′
∠ C 90	∠ B 33° 47′

Required the other parts.

EXAMPLE III.

In the plane triangle A B C,

Given	Ans.
B C 310	A C 281.73
∠ B 62° 9′	A B 209.95
∠ C 41° 13′	∠ A 76° 38′

Required the other parts.

EXAMPLE IV.

In the plane triangle A B C,

Given	Ans.
A B 217	∠ A 104° 47′
A C 305	∠ C 30° 12′
∠ B 45°	B C 417.03

Required the other parts.

EXAMPLE V.

On the plane triangle A B C,

Given	Ans.
A C 329	A B 305.69
B C 516	or 516.82
∠ B 37° 9′	∠ C 34° 8′
	or 71° 34′
	∠ A 108° 43′
	or 71° 17′

Required the other parts.

CASE II.

When two sides and their included angle are given, to find the rest.

RULE.

As the sum of any two sides of a plane triangle, is to their difference, so is the tangent of half the sum of their opposite angles, to the tangent of half their difference.

Then the half difference of these angles, added to their half sum, gives the greater angle, and subtracted from it gives the less.

And as all the angles are now known, the remaining side may be found by Case I.

Note. Half the sum of the two opposite, or unknown angles, is found by subtracting the given angle from 180°, and dividing the remainder by 2 (*l*).

(*l*) Instead of the tangent of half the sum of the two unknown angles, we may use the cotangent of half the given angle, or the tangent of half its supplement, which are all equal to each other.

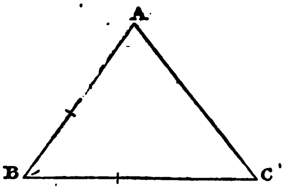
It may also be further observed with respect to the general solution above given, that when the triangle is isosceles, the angles at the base are each equal to the complement of half the given angle, or that at the vertex; and consequently, in this case, the third side may be found directly by Case I.

Also, if the angle included between the two given sides of the triangle be a right angle, or 90°, the third side, or hypothenuse, may be found by extracting the square root of the sum of the squares of the other two sides.

EXAMPLE.

In the plane triangle ABC,

$$\text{Given}\left\{\begin{array}{l} \left.\begin{array}{l}\text{AB } 1082 \\ \text{BC } 2384\end{array}\right\} \text{feet. Required the rest.} \\ \angle\text{B } 35^\circ\ 46' \end{array}\right.$$



BY CONSTRUCTION.

1. Take BC = 2384 from a scale of equal parts; and set off the ∠B = 35° 46′ by a scale of chords, or other instrument.

2. Make AB = 1082 by the same scale of equal parts, as before, and join A, C, and ABC will be the triangle required.

Then, the several parts being measured, we shall have ∠A = 121¼°, ∠C = 22¾°, and side AC = 1630 feet.

BY CALCULATION.

: AB + BC 3466 - - - - - 3.5398286

6.4601714

: AB ∼ BC 1302 - - - - - 3.1146110

:: Tan. $\frac{A+C}{2}$ 72° 7′ - - - - 10.4912414

: Tan. $\frac{A \sim C}{2}$ 49° 20′ - - - - 10.0660238

Sum 121° 27′ ∠A

Diff. 22° 47′ ∠C

Then,

: Sine ∠ A 121° 27′ or 58° 33′ - 9.9309978
0.0690022
: Side B C - - 2384 - - - - 3.3773063
: : Sine ∠ B - - 35° 46′ - - - 9.7667739
: Side A C - - 1630.3 - - - - 3.2130824

INSTRUMENTALLY.

In the first proportion, extend from 3466 (AB+BC), to 1302 (AB ∼ BC) on the line of numbers, and that extent will reach, on the tangents, from 72° 7′ (the contrary way, because the tangents are set back again from 45°) to beyond 45°; which being set so far back from 45°, falls upon 49$\frac{1}{3}$°, the fourth term.

In the second proportion, extend from 58$\frac{1}{2}$° (∠ A) to 35^{3}° (∠ B) on the sines, and that extent will reach, on the numbers, from 2384 (B C) to 1630, the fourth term, or side A C.

EXAMPLE II.

In the plane triangle A B C,

Given { $\overset{c}{AB}$ 907, BC 512, ∠ B 49° 10′ }   Ans. { ∠ C 96° 44′, ∠ A 34° 6′, $\overset{b}{AC}$ 691.02 }

Required the other parts.

EXAMPLE III.

In the plane triangle A B C,

Given { $\overset{c}{AB}$ 780, $\overset{a}{BC}$ 1004, ∠ B 90° }   Ans. { ∠ A 52° 9′, ∠ C 37° 51′, $\overset{b}{AC}$ 1271.4 }

Required the other parts.

EXAMPLE IV.

In the plane triangle A B C,

Given	A ^{c}B 530	Ans.	∠ A 80° 55′
	B ^{a}C 830		∠ C 39 5′
	∠ B 60°		A ^{b}C 727.94

Required the other parts.

EXAMPLE V.

In the plane triangle A B C,

Given	A ^{c}B 406	Ans.	∠ C 67° 45′
	B ^{a}C 359		∠ A 54° 55′
	∠ B 57° 20′		A ^{b}C 369.29

Required the other parts.

CASE III.

When the three sides are given, to find the angles.

RULE.

Make the longest side the base, and let fall a perpendicular upon it from the opposite angle.

Then, as the base, or sum of the segments, made by the perpendicular, is to the sum of the other two sides, so is the difference of those sides to the difference of the segments of the base.

And half this difference, added to half the base, will give the greater segment, or that next the greater side; and, subtracted from it, will give the less.

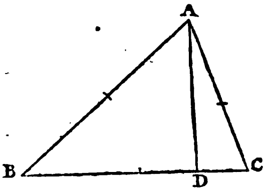
Whence, in each of the right-angled triangles, formed by the perpendicular, two sides and an angle opposite to one of them will be known; and consequently the other angles may be found, by Case I.

Or either of the angles may be found at one operation, by the logarithmic formula given in Case VI, of the Table for oblique-angled plane triangles, p. 33.

EXAMPLE I.

In the plane triangle A B C,

$$\text{Given}\left\{\begin{array}{l} AB\ 467 \\ AC\ 352 \\ BC\ 698 \end{array}\right\}\text{Yards. Required the angles.}$$



BY CONSTRUCTION.

Take B C = 698, by a scale of equal parts, and with the centres B, C, and radii 467 and 352, taken from the same scale, describe arcs intersecting each other in A; and join A B, A C, and A B C will be the triangle required.

Then, by measuring the angles with a protractor, or by the scale of chords, they will be found to be nearly as follows, viz. $\angle A = 116_{\tau}^{\circ}$, $\angle B = 27^{\circ}$, and $\angle C = 37^{\circ}$.

BY CALCULATION.

Having let fall the perpendicular A D, it will be

: B C or B D + D C 698 - - - $\underline{2.8438554}$
$\quad$ 7.1561446
: A B + A C - - 819 - - - 2.9132839
: : A B ∼ A C - - 115 - - - $\underline{2.0606978}$
: B D ∼ D C - - 134.93 - - $\underline{2.1301263}$

$$\text{Hence } \frac{698+134.93}{2} = 416.46 = BD$$

$$\text{And } \frac{698-134.93}{2} = 281.53 = CD$$

Then, in the triangle ABD, right $\angle^{d}$ at D,

: AB - - - - - - - 467 - - - - 2.6693169
: BD - - - - - 416.46 - - - 2.6195733
:: Sine ∠D - - - 90° - - - - 10.0000000
: Sine ∠BAD - 63° 6′ - - - 9.9502564
90° 0′
26° 54′ ∠B

And, in the triangle ACD, right $\angle^{d}$ at D,

: AC - - - - - 352 - - - - 2.5465427
: DC - - - - - 281.53 - - - 2.4495247
:: Sine ∠D - - - 90° - - - - 10.0000000
: Sine ∠CAD - 53° 6′ - - - 9.9029820
90° 0′
36° 54′ ∠C
Also 63° 6′ ∠BAD
And 53° 6′ ∠CAD
Makes 116° 12′ ∠BAC

Whence ∠B=26° 54′, ∠C=36° 54′, and ∠BAC=116° 12′.

INSTRUMENTALLY.

In the first proportion, extend from 698 (BC) to 819 (AB+AC) on the line of numbers, and that extent will reach, on the same line, from 115 (AB∼AC) to 135, the difference of the segments of the base (BD∼DC).

In the second proportion, extend from 467 (AB) to 416 (BD) on the numbers, and this extent will reach, on the sines, from 90° (∠D) to $63^\circ\frac{1}{10}$ (∠BAD).

In the third proportion, extend from 352 (AC) to 281½ (DC) on the numbers, and that extent will reach, on the sines, from 90° (∠D) to $53^\circ\frac{1}{10}$ (∠CAD).

EXAMPLE II.

In the plane triangle A B C,

Given	A^{c}B 750	Ans.	∠ A 50° 2′
	A^{b}C 340		∠ B 26° 7′
	B^{a}C 592		∠ C 103° 55′

Required the angles.

EXAMPLE III.

In the plane triangle A B C,

Given	A^{c}B 500	Ans.	∠ A 53° 8′
	A^{b}C 300		∠ B 36° 52′
	B^{a}C 400		∠ C 90°

Required the angles.

EXAMPLE IV.

In the plane triangle A B C,

Given	A^{c}B 760	Ans.	∠ A 69° 3′
	A^{b}C 429		∠ B 33° 27′
	B^{a}C 727		∠ C 77° 30′

Required the angles.

EXAMPLE V.

In the plane triangle A B C,

Given	A^{c}B 1002	Ans.	∠ A 133° 46′
	A^{b}C 1960		∠ B 30° 59′
	B^{a}C 2750		∠ C 15° 15′

Required the angles.

EXAMPLE VI.

In the plane triangle A B C,

Given	A^{c}B 169	Ans.	∠ A 90° 29′
	A^{b}C 169		∠ B 44° 46′
	B^{a}C 240		∠ C 44° 46′

Required the angles.

These three problems include all the cases or varieties of plane triangles, as well right-angled as oblique, that can possibly happen; but the following theorem, for right-angled triangles, will sometimes be found more convenient in practice, than the analogy which is furnished by the general rule.

THEOREM, OR CASE IV.

In any right-angled triangle, As radius is to the tangent of either of the acute angles, so is the leg adjacent to that angle to the leg opposite to it; and vice versâ.

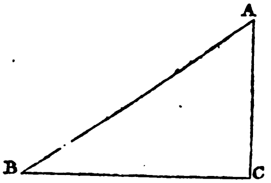
Or, As radius is to the cotangent of either of the acute angles, so is the leg opposite to that angle to the leg adjacent to it; and vice versâ (*m*).

It may also be observed, that as the sine of either of the acute angles of a right-angled triangle is equal to the cosine of the other, the latter may be used instead of the former, whenever it renders the operation more simple.

EXAMPLE I.

In the right-angled plane triangle A B C,

Given $\left\{\begin{array}{l}\text{B C } 327 \\ \angle \text{B } 54^\circ\ 17'\end{array}\right\}$ Required the other parts.



(*m*) To this we might also have added the following analogy: As radius is to the secant of either of the acute angles, so is the leg

BY CONSTRUCTION.

Make B C = 327, and ∠ B = 54° 17′; then raise the perpendicular C A, meeting B A in A; and A B C will be the triangle required: in which A B is found to measure 560, A C 454¾, and ∠ A, which is the complement of ∠ B, 35½°.

BY CALCULATION.

:	Rad. or sine - - -	90° - - -	10.0000000
:	Tan. ∠ B - - - - - -	54° 17′ -	10.1432626
: :	Side B C - - - - - -	327 - -	2.5145478
:	Side A C - - - - - -	454.79 -	2.6578104
:	Sine ∠ A or cos. ∠ B	54° 17′ -	9.7662473
:	Side B C - - - - - -	327 - -	2.5145478
: :	Rad. or sine ∠ C -	90° - - -	10.0000000
:	Side A B - - - - - -	560.14 -	2.7483005

And 90° − 54° 17′ = 35° 43′ ∠ A.

INSTRUMENTALLY.

Extend the compasses from 45° to 54¼° (∠ B) on the tangents, and that extent will reach from 327 (B C) to 454½, on the line of numbers, for the side A C.

And the extent from 54¼° (∠ A) to 90° on the

---

adjacent to that angle to the hypothenuse. Or, As radius is to the cosecant of either of the acute angles, so is the leg opposite to that angle to the hypothenuse.

But the rule for this case is as readily performed by the sines and cosines, which are always to be found in the logarithmic tables, where the secant is frequently omitted.

sines, will reach from 327 (B C) to 560, on the line of numbers, for the hypothenuse A B.

EXAMPLE II.

In the right-angled triangle A B C,

$$\text{Given}\left\{\begin{array}{l} \text{B C } 520 \\ \angle \text{A } 43^\circ\ 21' \end{array}\right. \qquad \text{Ans.}\left\{\begin{array}{l} \text{A C } 550.85 \\ \text{A B } 757.52 \\ \angle \text{B } 46^\circ\ 39' \end{array}\right.$$

To find the other sides and angle.

EXAMPLE III.

In the right-angled triangle A B C,

$$\text{Given}\left\{\begin{array}{l} \text{A C } 307 \\ \text{B C } 421 \end{array}\right. \qquad \text{Ans.}\left\{\begin{array}{l} \angle \text{B } 36^\circ\ 6' \\ \angle \text{A } 53^\circ\ 54' \\ \text{A B } 521.05 \end{array}\right.$$

To find the other sides and angles.

EXAMPLE IV.

In the right-angled triangle A B C,

$$\text{Given}\left\{\begin{array}{l} \text{A C } 490 \\ \angle \text{A } 51^\circ\ 7' \end{array}\right. \qquad \text{Ans.}\left\{\begin{array}{l} \text{B C } 607.62 \\ \text{A B } 780.58 \\ \angle \text{B } 38^\circ\ 53' \end{array}\right.$$

To find the other sides and angle.

To these rules it has been judged necessary to add the following tables, which contain the solutions of all the cases of plane trigonometry before given; together with such additional formulæ, for the tangents, as are better adapted, in certain instances, to the producing of accurate results than those derived from the sines and cosines in the former analogies.

The reason of this deficiency of the sines and cosines, in the cases alluded to, is, that if an arc near

90° is to be found by means of its sine; or a very small arc, or one near 180°, by means of its cosine, the variation of these lines is so small, that they will not change in the tables for many seconds.

Thus, if the log-sine, or log-cosine, of a required arc should come out 9.9999998, this number, in the tables, is the sine of an arc from 89° 56′ 19″ to 89° 57′ 8″ or the cosine of an arc from 2′ 52″ to 3′ 41″; and consequently it is impossible to say what arc or angle, between these limits, is to be taken, owing to the tables not being continued to more than seven places of decimals.

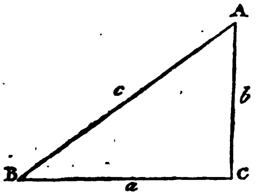
An error also, of an opposite kind, will arise in finding arcs near 90° by their tangents, or those near 0° by their cotangents, as the largeness of the logarithmic differences, in this part of the quadrant, will render the method of determining them by proportional parts, incorrect, when they are to be found to seconds, by the common tables, or to parts of seconds, by Taylor's tables.

But in all other cases, the magnitude and great variation of the logarithmic differences render the use of the tangents preferable to that of the sines or cosines; besides which, the above error, in the use of the common tables, may always be avoided, by subtracting the log-tangent, or log-cotangent, from 20, and then finding the corresponding arc, or angle, in the first part of those tables, where it is usually given to seconds for the first two or three degrees of the quadrant.

To this we may further add, that when a sine, cosine, &c of the kind here mentioned enters into the calculation, as one of the data, it is rather favourable than otherwise to the accuracy of the result, or the value of the thing sought; as any small error in the given arc, or angle, will not much affect the tabular value of its sine or cosine.

Note. C L is used, in the following formulæ, to denote the co-log, or the complement of the common tabular logarithm of the number answering to the letter or expression to which it is prefixed. And the sign ~ expresses the difference of the two quantities between which it is placed, when it is not known which of them is the greater.

SOLUTIONS OF ALL THE CASES OF RIGHT-ANGLED PLANE TRIANGLES.



I. Given the hypothenuse and either of the oblique angles, to find either of the legs.

RULE.

$$\text{As rad} : \text{hyp} :: \left\{\begin{matrix}\text{sin given } \angle \\ \text{or} \\ \text{cos given } \angle\end{matrix}\right. : \left\{\begin{matrix}\text{its opp. leg} \\ \\ \text{its adj}^{\text{t}}\text{. leg}\end{matrix}\right.$$

Or,

$$a = \frac{c \sin. A}{r} \text{ or } \frac{c \cos. B}{r};\ b = \frac{c \sin. B}{r} \text{ or } \frac{c \cos. A}{r}$$

$\text{L}a = \text{L}c + \text{L}\sin \text{A}$ (or $\text{L}\cos \text{B}$) $- 10$; $\text{L}b = \text{L}c + \text{L}\sin \text{B}$ (or $\text{L}\cos \text{A}$) $- 10$.

$$\text{Area}\ \Delta = \frac{c^2 \sin. 2\text{A}}{4r} \text{ or } \frac{c^2 \sin 2\text{B}}{4r}.$$

$\text{L area} = 2\text{L}c + \text{L}\sin 2\text{A} - 10.6020600$; or $2\text{L}c + \text{L}\sin 2\text{B} - 10.6020600$.

II. Given the hypothenuse and either of the legs, to find either of the oblique angles.

RULE I.

$$\text{As hyp : rad : : given leg : } \begin{cases} \text{sin its opp.} \angle \\ \text{or} \\ \text{cos its adj}^{\text{t}}. \angle \end{cases}$$

Or,

$$\text{Sin A (or cos B)} = \frac{ra}{c};\ \text{sin B (or cos A)} = \frac{rb}{c}.$$

$\text{L}\sin \text{A}$ (or $\text{L}\cos \text{B}$) $= \text{€}\,\text{L}c + \text{L}a$; $\text{L}\sin \text{B}$ (or $\text{L}\cos \text{A}$) $= \text{€}\,\text{L}c + \text{L}b$.

If A be near 90°, find ∠ B, which is its complement, and the contrary when ∠ B is near 90°.

RULE II.

$$\text{Tan}\ \tfrac{1}{2}\text{A} = r\sqrt{\frac{c-b}{c+b}};\ \tan \tfrac{1}{2}\text{B} = r\sqrt{\frac{c-a}{c+a}}.$$

$$\text{L}\tan \tfrac{1}{2}\text{A} = \frac{\text{€}\,\text{L}(c+b) + \text{L}(c-b) + 10}{2};$$

$$\text{L}\tan \tfrac{1}{2}\text{B} = \frac{\text{€}\,\text{L}(c+a) + \text{L}(c-a) + 10}{2}.$$

Where ½ A, or ½ B, is always less than 45°.

$$\text{Area}\ \Delta = \frac{a}{2}\sqrt{(c+a)(c-a)} \text{ or } \frac{b}{2}\sqrt{(c+b)(c-b)}.$$

$\text{L area} = \tfrac{1}{2}\{\text{L}(c+a) + \text{L}(c-a)\} + \text{L}a - .3010300$; or $\tfrac{1}{2}\{\text{L}(c+b) + \text{L}(c-b)\} + \text{L}b - .3010300$.

III. Given the hypothenuse and either of the legs, to find the other leg.

RULE I.

Find either of the oblique $\angle^{s}$ by Case II.; and then the required leg by Case I.

RULE II.

$$a = \sqrt{(c+b) \times (c-b)};\quad b = \sqrt{(c+a) \times (c-a)}$$

$$\text{L}a = \frac{\text{L}\,(c+b)+\text{L}\,(c-b)}{2};\quad \text{L}b = \frac{\text{L}\,(c+a)+\text{L}\,(c-a)}{2}.$$

IV. Given either of the legs and either of the oblique angles, to find the other leg.

RULE.

$$\text{As rad : given leg} :: \left\{\begin{array}{l}\tan\angle\ \text{adj}^{t}.\ \text{given leg}\\ \qquad\quad \text{or}\\ \cot\angle\ \text{opp. given leg}\end{array}\right. : \text{req}^{d}.\ \text{leg}.$$

Or,

$$a = \frac{b \tan \text{A}}{r} \text{ or } \frac{b \cot \text{B}}{r};\quad b = \frac{a \tan \text{B}}{r} \text{ or } \frac{a \cot \text{A}}{r}.$$

$\text{L}a = \text{L}b + \text{L} \tan \text{A}$ (or $\text{L} \cot \text{B}$) $- 10$; $\text{L}b = \text{L}a +$ $\text{L} \tan \text{B}$ (or $\text{L} \cot \text{A}$) $- 10$.

$$\text{Area } \Delta = \frac{b^2 \tan \text{A}}{2r} \text{ or } \frac{a^2 \tan \text{B}}{2r}.$$

L area $= 2\text{L}b + \text{L} \tan \text{A} - 10.3010300$; or $2\text{L}\,a + \text{L} \tan \text{B} - 10.3010300$.

V. Given either of the legs and either of the oblique angles, to find the hypothenuse.

RULE.

$$\left.\begin{array}{l}\text{As sin} \angle \text{ opp}^t\text{. given leg} \\ \text{Or cos} \angle \text{ adj}^t\text{. given leg}\end{array}\right\} : \text{given leg} : : \text{rad} : \text{hyp.}$$

Or,

$$c = \frac{r\,a}{\sin A \text{ (or } \cos B)} = \frac{r\,b}{\sin B \text{ (or } \cos A)}$$

L c = € L sin A (or € L cos B) + L a = € L sin B (or € L cos A) + L b.

VI. Given the two legs, to find either of the oblique angles.

RULE.

$$\text{As either leg} : \text{rad} : : \text{other leg} : \left\{\begin{array}{c}\text{tan its opp.} \angle \\ \text{or} \\ \text{cot its adj}^t\text{.} \angle\end{array}\right.$$

Or,

$$\text{Tan } A \text{ (or cot } B) = \frac{r\,a}{b};\ \tan B \text{ (or cot } A) = \frac{r\,b}{a}.$$

L tan A (or L cot B) = € L b + L a; L tan B (or L cot A) = € L a + L b.

If ∠A be near 90°, find ∠B, which is its complement; and the contrary when ∠B is near 90°.

Area $\triangle = \frac{1}{2}\,a\,b$; L area = L a + L b − .3010300.

VII. Given the two legs, to find the hypothenuse.

RULE I.

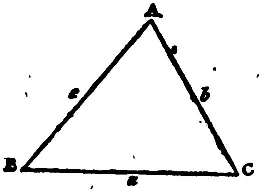
Find either of the oblique ∠^{s} by Case VI. and then find the hypothenuse by Case V.

RULE II.

$$c = \sqrt{a^2+b^2} = a\sqrt{\left(1+\frac{b^2}{a^2}\right)} = b\sqrt{\left(1+\frac{a^2}{b^2}\right)}$$

which formulæ do not admit of convenient logarithmic expressions.

SOLUTIONS OF ALL THE CASES OF OBLIQUE-ANGLED PLANE TRIANGLES.



I. Given a side and two angles, to find either of the other two sides.

RULE.

Find the third, or remaining ∠, if necessary, by subtracting the sum of the two given ∠ˢ from 180°.

Then,

As sin ∠ opp. given side : given side : : sin ∠ opp. required side : required side.

Or,

$$a = \frac{b \sin A}{\sin B}$$

$$\text{L}a = \text{C L} \sin B + \text{L} \sin A + \text{L} b - 10.$$

Either of the other sides may also be expressed in the same form, by taking the side and angles which are similarly situated with respect to the side whose value is sought. And the same may be observed in all the other cases, where only one side or an angle is exhibited by the formula.

$$\text{Area } \Delta = \frac{b^2 \sin \text{A}}{2r \sin \text{B}} \sin(\text{A}+\text{B}).$$

$\text{L area} = \text{C L} \sin \text{B} + \text{L} \sin(\text{A}+\text{B}) + \text{L} \sin \text{A} + 2 \text{L} b - 20.3010300.$

II. Given two sides and an angle opposite to one of them, to find the other angles.

RULE I.

As side opp. given ∠ : sin given ∠ : : other side : sin its opp. ∠.

Which ∠ is acute, if it be opp. to the least of the two given sides; but, if it be opp. to the greater, it may be either an acute ∠ or its sup^{t}. or else a right angle.

The 3d, or remaining ∠ = 180° — sum of the other two ∠^{s}.

Or,

$$\text{Sin A} = \frac{a \sin \text{B}}{b}$$

$$\text{L} \sin \text{A} = \text{C L} b + \text{L} a + \text{L} \sin \text{B} - 10.$$

RULE II.

$$\text{Let C L} b + \text{L} a + \text{L} \sin \text{B} - 10 = \text{L} \tan \varphi.$$

$$\text{Then, L} \tan (45^\circ \sim \tfrac{1}{2}\text{A}) = \frac{10 + \text{L} \tan (45^\circ \sim \varphi)}{2}.$$

Where arc φ can never exceed 45°; and ∠ A, which is found by subtracting 2 $(45^\circ \sim \frac{1}{2}\text{A})$ from 90°, is subject to the same ambiguity as in rule I.

Note. In this and the following case, the given sides and ∠ must be so taken, that the result, found by rule I, or the 1st part of rule II, shall not be greater than radius, otherwise the Δ is impossible.

III. Given two sides and an angle opposite to one of them, to find the remaining side.

RULE.

Find the other two ∠s by case II, observing that they will be equally subject to the ambiguity there mentioned; and then find the remaining side by case I.

Or, the side, and area, may be found by the following formulæ:

$$c = \frac{b \cos \mathrm{A} \pm \sqrt{r^2 a^2 - b^2 \sin^2 \mathrm{A}}}{r}$$

$$\text{Area}\,\Delta = \frac{b \sin \mathrm{A}\,(b \cos \mathrm{A} \pm \sqrt{r^2 a^2 - b^2 \sin^2 \mathrm{A}})}{2\,r^2}.$$

But neither of these admit of convenient logarithmic expressions.

If the triangle be isosceles, or have $a = b$, the rule will give $c = \frac{2\,a}{r} \cos \mathrm{A}$, or $\mathrm{L}\,c = \mathrm{L}\,a + \mathrm{L} \cos \mathrm{A} - 9.6989700$; and the area of the $\Delta = \frac{a^2}{2r} \sin 2\,\mathrm{A}$, or L area $= 2\mathrm{L}a + \mathrm{L} \sin 2\mathrm{A} - 10.3010300$.

IV. Given two sides and their included angle, to find the other two angles.

RULE.

As sum of the two given sides : their difference : : cot ½ included ∠ : tan ½ difference other two ∠s.

Which ½ difference, added to the complement of ½ the given ∠, gives the ∠ opposite the greater of the two given sides; and, subtracted from it, gives the ∠ opposite the less.

Or,

$$\text{Tan}\ \tfrac{1}{2}\,(\text{B}\sim\text{C}) = \frac{b\sim c}{b+c}\cot\tfrac{1}{2}\,\text{A}.$$

$\text{L}\tan\frac{1}{2}\,(\text{B}\sim\text{C}) = €\,\text{L}\,(b+c) + \text{L}\,(b\sim c) + \text{L}\cot\frac{1}{2}\,\text{A} - 10.$

Then, $(90° - \frac{1}{2}\,\text{A}) + \frac{1}{2}\,(\text{B}\sim\text{C})$ = ∠ opp. greater side; and $(90° - \frac{1}{2}\,\text{A}) - \frac{1}{2}\,(\text{B}\sim\text{C})$ = ∠ opp. less side.

If the values of b, c, in this case, should be given in logarithms, instead of the natural numbers, as is sometimes the case in astronomical calculations, the following formulæ will be found more convenient in practice than that given above.

Let € L greater side + L less side = L tan φ; in which case arc (φ) is always less than 45°.

Then,

$\text{L}\tan\frac{1}{2}\,(\text{B}\sim\text{C}) = \text{L}\cot\frac{1}{2}\,\text{A} + \text{L}\tan(45° - \varphi) - 10.$
Where $(90° - \frac{1}{2}\,\text{A}) + \frac{1}{2}\,(\text{B}\sim\text{C})$ = ∠ opp. greater side, and $(90° - \frac{1}{2}\,\text{A}) - \frac{1}{2}\,(\text{B}\sim\text{C})$ = ∠ opp. less side, as before.

Or, either of the two ∠ˢ may be found by the following formulæ:

$$\text{Tan B} = \frac{r\sin\text{A}}{r\left(\frac{c}{b}\right) - \cos\text{A}};\quad \tan\text{C} = \frac{r\sin\text{A}}{r\left(\frac{b}{c}\right) - \cos\text{A}}$$

$$\text{Sin B} = \frac{b\sin\text{A}}{\sqrt{b^2 + c^2 - \frac{2bc}{r}\cos\text{A}}};\quad \sin\text{C} = \frac{c\sin\text{A}}{\sqrt{b^2 + c^2 - \frac{2bc}{r}\cos\text{A}}}$$

But these do not admit of convenient logarithmic expressions.

$$\text{Area } \triangle = \frac{b\,c \sin \text{A}}{2r}.$$

$$\text{L area} = \text{L } b + \text{L } c + \text{L} \sin \text{A} - 10.3010300.$$

V. Given two sides and their included angle, to find the other side.

RULE.

Find the other two ∠ˢ by case IV; and then the remaining side by case I.

Or, the side may be found by the following formulæ:

$$c = \sqrt{(a \sim b)^2 + \frac{4ab}{r^2} \sin^2 \tfrac{1}{2}\text{C}} = \frac{1}{r} \sqrt{(a+b)^2 \sin^2 \tfrac{1}{2}\text{C} + (a \sim b)^2 \cos^2 \tfrac{1}{2}\text{C}}.$$

Or,

$$c = \sqrt{(a+b)^2 - \frac{4ab}{r^2} \cos^2 \tfrac{1}{2}\text{C}} = \sqrt{a^2 + b^2 - \frac{2ab}{r} \cos \text{C}}.$$

But these do not admit of convenient logarithmic expressions.

If the △ be isosceles, or have $a = b$, the rule will give $c = \frac{2\,a \sin \frac{1}{2}\text{C}}{r}$, or $\text{L } c = \text{L } a + \text{L} \sin \tfrac{1}{2}\text{C} - 9.6989700$.

VI. Given the three sides, to find either of the angles.

RULE I.

As the longest side, taken as a base : sum of the other two sides : : difference of those sides : difference of the segments of the base, made by a perpendicular from the opposite ∠.

Then, ½ this difference added to ½ the base, gives the greater segment, or that next the greater side; and subtracted from it, gives the less.

And as the perpendicular divides the △ into two right-angled △ˢ, in each of which the hypothenuse and a leg are known, the angles may be found by case II. of right-angled triangles.

RULE II.

$$\text{Tan}\ \tfrac{1}{2}\,\mathrm{A} = r\sqrt{\frac{(\frac{1}{2}s-b)\times(\frac{1}{2}s-c)}{\frac{1}{2}s\times(\frac{1}{2}s-a)}}.$$

$$\mathrm{L}\ \tan\tfrac{1}{2}\,\mathrm{A} = \frac{\mathrm{C}\,\mathrm{L}\,\frac{1}{2}s + \mathrm{C}\,\mathrm{L}\,(\frac{1}{2}s-a) + \mathrm{L}\,(\frac{1}{2}s-b) + \mathrm{L}\,(\frac{1}{2}s-c)}{2}$$

Where s denotes the sum of the three sides $a+b+c$; and $\frac{1}{2}\ \angle\,\mathrm{A}$ is always acute.

Or, the latter of these two formulæ may be expressed in words at length, as follows:

Add together the log. of ½ the sum of the three sides, and the log. of the difference between this ½ sum and the side opposite the ∠ sought, and find the complement of their sum, by subtracting the index from 19, and the rest of the figures from 9, as usual.

Then to this complement add the logarithms of the differences between the said ½ sum and each of the other two sides, and the result, divided by 2, will give the tangent of ½ the required angle.

$$\text{Area}\ \triangle = \tfrac{1}{4}\sqrt{\overline{(b+c)^2-a^2}\times\overline{a^2-(b\sim c)^2}};$$

$$\text{Or,}\ \sqrt{\tfrac{1}{2}s\times(\tfrac{1}{2}s-a)\times(\tfrac{1}{2}s-b)\times(\tfrac{1}{2}s-c)}.$$

$$\mathrm{L}\ \text{area} = \tfrac{1}{2}\left\{\mathrm{L}\,\tfrac{1}{2}s + \mathrm{L}\,(\tfrac{1}{2}s-a) + \mathrm{L}\,(\tfrac{1}{2}s-b) + \mathrm{L}\,(\tfrac{1}{2}s-c)\right\}$$

To the formula above given, we may also add the following:

$$\text{Sin}\,\tfrac{1}{2}\,\mathrm{A} = r\sqrt{\frac{(\frac{1}{2}s - b) \times (\frac{1}{2}s - c)}{b\,c}};\ \cos\tfrac{1}{2}\,\mathrm{A} = r\sqrt{\frac{\frac{1}{2}s\,(\frac{1}{2}s - a)}{b\,c}}.$$

Or,

$$\mathrm{L}\sin\tfrac{1}{2}\,\mathrm{A} = \frac{\text{€}\,\mathrm{L}\,b + \text{€}\,\mathrm{L}\,c + \mathrm{L}\,(\frac{1}{2}s - b) + \mathrm{L}\,(\frac{1}{2}s - c)}{2}.$$

$$\mathrm{L}\cos\tfrac{1}{2}\,\mathrm{A} = \frac{\text{€}\,\mathrm{L}\,b + \text{€}\,\mathrm{L}\,c + \mathrm{L}\,\frac{1}{2}s + \mathrm{L}\,(\frac{1}{2}s - a)}{2}.$$

The former of which, or that for the sine, must be used when $\frac{1}{2}$ A differs considerably from 90°, and the latter when it is near 90°.

It may likewise be observed, that when the Δ is isosceles, or has $b = c$, we shall have

$$\text{Sin}\,\tfrac{1}{2}\,\mathrm{A} = \frac{r\,a}{2\,b};\ \cos\tfrac{1}{2}\,\mathrm{A} = \frac{r}{2\,b}\sqrt{(2\,b + a) \times (2\,b - a)}$$

Or, $\mathrm{L}\sin\frac{1}{2}\,\mathrm{A} = \text{€}\,\mathrm{L}\,b + \mathrm{L}\,a - .301300$, and $\mathrm{L}\cos\frac{1}{2}\,\mathrm{A} = \frac{1}{2}\,\text{€}\,\mathrm{L}\,2\,b + \frac{1}{2}\left\{\mathrm{L}\,(2\,b + a) + \mathrm{L}\,(2\,b - a)\right\}$

And if the Δ be equilateral, having each of its sides $= a$, we shall have area $\Delta = \frac{1}{4}\,a^2\sqrt{3}$; or L area $= 2\,\mathrm{L}\,a - .3634993$.

We may here also subjoin the two following formulæ:

$$\text{Sin}\,\mathrm{A} = r\sqrt{\frac{a^2 - (b \sim c)^2}{2\,b\,c}};\ \cos\mathrm{A} = r\left(\frac{b^2 + c^2 - a^2}{2\,b\,c}\right),$$

which have been found highly useful in many trigonometrical investigations.

MISCELLANEOUS EXAMPLES.

1. How many inches does an angle of 1″ subtend at the distance of six miles; supposing the object to be perpendicular to the horizon?

Ans. 1¾ inch nearly.

2. The hypothenuse of a right-angled triangle being 13 feet 10 inches, and the base 9 feet 7 inches, it is required to find the perpendicular.

Ans. 9 feet 11$\frac{7}{10}$ inches.

3. The hypothenuse of a right-angled triangle bebeing 6395 feet, and one of the acute angles 39° 50′ 38″, it is required to find the two legs.

Ans. 4097.262 and 4910.035

4. The hypothenuse of a right-angled triangle being 19630040, and one of the legs 19630000, it is required to find the two acute angles.

Ans. 6′ 56″$\frac{2}{5}$ and 89° 53′ 3″$\frac{3}{5}$

5. If the base of an oblique-angled plane triangle be 60, and the other two sides 30 and 40, what is its perpendicular altitude? Ans. 17.77561

6. If the base of a plane triangle be 60, and the other two sides 30 and 40, what are the lengths of the segments of the base, made by a line bisecting the vertical angle? Ans. 34.28571 and 25.71428

7. Supposing one angle of a plane triangle to be 139° 54′, and the two sides about that angle in the ratio of 5 to 9, it is required to find the other two angles. Ans. 26° 0′ 10″$\frac{3}{5}$ and 14° 5′ 49″$\frac{2}{5}$

8. The sides of a plane triangle being 14373, 13172, and 11845, it is required to find the three angles.

Ans. 69° 54′ 10″, 59° 23′ 19″, and 50° 42′ 31″

9. If the three angles of a plane triangle be 104° 42′ 15″, 49° 24′ 5″, and 25° 53′ 40″, and the side opposite the greatest angle 476.75 yards, what are the other two sides? Ans. 215.2533 and 374.2469

10. If the sides of a plane triangle be in proportion to each other as $\frac{1}{5}$, $\frac{1}{7}$, and $\frac{1}{9}$, what are the angles?

Ans. $32° 44' 39''$, $44° 3' 41''$, and $103° 11' 40''$

11. In a right-angled plane triangle, the three sides are 3, 4, and 5, from which it is required to find the angles. Ans. $36° 52' 11''\frac{3}{4}$, $53° 7' 48''\frac{1}{4}$, and $90°$

12. In an oblique-angled plane triangle, the three sides are 4, 5, and 6, what are the three angles?

Ans. $41° 24' 34''\frac{5}{10}$, $55° 46' 16'\frac{3}{10}$, and $82° 49' 9''\frac{2}{10}$

13. There are three towns, A, B, and C, the distance of A from B is 5 miles, of B from C 9 miles, and of C from A 7 miles, what are their respective bearings from each other? Ans. $\angle$ A $95° 44' 21''$, $\angle$ C $33° 33' 26''$, $\angle$ B $50° 42' 13''$

14. How must three trees, A, B, C, be planted, so that the angle at A may be double that at B, and the angle at B double that at C; and that a line of 100 yards may go just round them?

Ans. A B 19.80621, A C 35.68958, and B C 44.50418

OF THE MENSURATION OF HEIGHTS AND DISTANCES.

The measuring of heights and distances depends upon the use of certain instruments for taking angles, and the rules of Plane Trigonometry, already delivered; which being separately or jointly applied, as the case may require, will resolve most questions of this kind that can occur in practice (*p*).

(*p*) Horizontal and vertical angles are usually taken with a theodolite furnished with one or two telescopes, and a vertical

In addition, however, to the properties of plane triangles, given in page 11, it may be necessary to lay down a few others relating to angles, parallel lines, &c. which in several instances will be found of great use in facilitating both the constructions and calculations.

1. The two angles, which are made by one right line meeting another, are together equal to two right angles, or 180°.

2. If two right lines intersect each other, the vertical or opposite angles will be equal.

3. If a right line intersects two parallel right lines, it makes the alternate angles equal, and the outward angle equal to the inward opposite angle, on the same side.

4. If one side of a plane triangle be produced, the

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arc; and if the circles of the instrument are about $3\frac{1}{2}$ inches radius, the observed angles may be read off to half a minute. But if the angles are oblique to the horizon, they must be taken with a sextant, or Hadley's quadrant, which is to be held in such a position that its plane may pass through both objects and the eye of the observer; and when altitudes are determined by this instrument, it is done by reflecting the object from an artificial horizon.

Short bases, for temporary use only, are commonly measured with rods, or Gunter's chain, of 66 feet in length; but the common 50 or 100 feet tapes are better adapted for expedition. With these lines, when the ground is tolerably level, and the direction, or *alignement*, of the base pretty correct, the error in distance will probably be about 3 inches in 50 feet, or $\frac{1}{200}$ of the whole measurement, as long as the tapes are kept dry.

outward angle will be equal to the sum of the two inward opposite angles.

5. All angles in the same segment of a circle, or which stand upon the same arc, are equal.

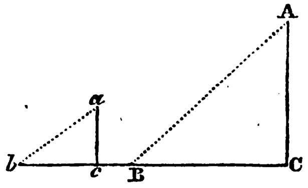
6. An angle at the centre of a circle is double that at the circumference, when they both stand on the same arc.

7. An angle in a semicircle, or that which stands upon half the circumference, is a right angle, or 90°.

8. If a right line be drawn parallel to one of the sides of a plane triangle, it will cut the other two sides proportionally.

It may also be remarked, that some of the simplest cases of heights and distances, may be resolved without the assistance of trigonometry, or of any instrument for taking angles, by one or other of the following methods:

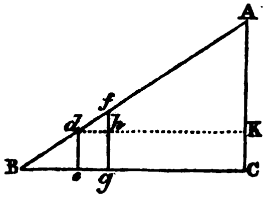
1. By the property of similar triangles; from which it is known that objects are in proportion to each other as the lengths of their shadows.



Thus, if the height of the pole *a c* be 10 feet, the length of its shadow *c b* 8 feet, and the shadow C B, of the object A C, 50 feet:

Then 8 (*c b*) : 10 (*a c*) : : 50 (C B) : 62½ feet = height A C.

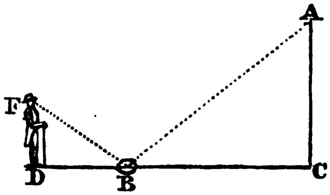
2. Another method is by means of two poles of unequal lengths, set up parallel to the object, so that the observer may see the top of the object over the tops of both the poles.



Thus, let the length of the pole de be 6 feet, that of the pole fg 8 feet, their distance asunder eg 10 feet, and the distance eC, of the shorter pole from the object, 190 feet.

Then the triangles dhf and dKA being similar, dh or $eg : hf : : d$K or eC : KA; or $10 : 8 - 6 : : 190 : 38$ feet $=$ AK. Hence AK $+$ KC $=$ AK $+ de = 38 + 6 = 44 =$ AC.

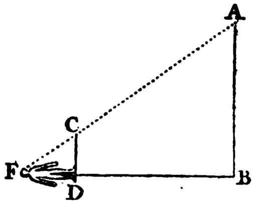
3. A third method, is by viewing the image of the top of the object reflected from some smooth surface, as a mirror placed horizontally, a vessel of water, &c.



Thus, let B be the reflecting surface, at the distance of 94 feet from the bottom of the object AC; and let a person at D, 6 feet from B, with his eye 5½ feet above the ground, view the image of the top of the object at B.

Then, because, from the principles of optics, the angle of incidence A B C, is equal to the angle of reflection F B D, the triangles B D F, B C A, will be similar; and consequently B D : D F : : B C : C A; or 6 : $5\frac{1}{2}$: : 94 : $86\frac{1}{6}$ feet = A C.

4. A fourth method, is for the observer to fix a pole upright in the ground, by trials, so that having laid himself on his back, with his feet against the bottom of it, he may see the top of the pole and that of the object in the same right line.



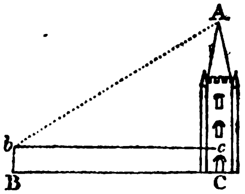
In which case the distance F D from the foot of the pole to the eye of the observer, will be in proportion to the height of the pole C D, as the whole distance F B is to the height of the object A B.

And if the height of the pole C D be equal in length to the observer F D, the distance F B will be equal to the height of the object A B.

PRACTICAL QUESTIONS.

1. Having measured a distance of 220 feet, in a direct horizontal line, from the bottom of a steeple, I then found the angle of elevation of its top to be 46° 30′; required the height of the steeple (*q*).

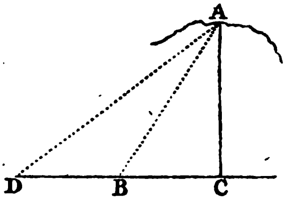
(*q*) In Mauduit's Trigonometry (Crakelt's Trans. p. 182.) it is shown that the error of any altitude A *c* is to the error committed



As rad or sine - - -	90° - - - -	10.0000000
Is to *b c* - - - - - -	220 feet -	2.3424227
So is tan ∠ A *b c* -	46° 30′ - -	10.0227500
To height A *c* - - -	231.83 feet	2.3651727

which added to the height of the instrument B *b*, will give the whole height A C.

2. It is required to find the perpendicular height of a hill, the angle of elevation of which, taken at the bottom, was 46° 10′, and 110 yards further off, on a level with the bottom, and in the same vertical plane with the former, 29° 56′.



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in taking the ∠ A *b c*, as double the height A *c* is to the sine of double the observed ∠ A *b c*. Whence the error that may arise in taking the altitude of an object, by a sextant, or theodolite, will be the least possible when the sine of double the observed ∠ is the greatest possible; which is when it is 45°. So that in finding altitudes, the observed ∠ should be taken as near 45° as can conveniently be done. At an exact altitude of 45°, if an error of 1′ be made in the determination of the observed ∠, the error in altitude will be $\frac{1}{1719}$ part of the whole; and if the observed ∠ be greater or less than 45°, the error in altitude will be increased in the ratio of radius to the sine of double the said ∠.

∠ A B C - - - - - 46°
∠ A D C - - - - - 31°
∠ D A B - - - - - 15°

As sin ∠ D A B - - 16° 14′ - - - 9.4464591
Is to D B - - - - - 110 - - - - 2.0413927
So is sin ∠ D - - - 29° 56′ - - - 9.6980936

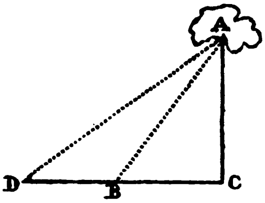
To A B - - - - - - - - - - - - - 2.2929272

Then,

As rad, or sin - - 90° - - - - 10.0000000
Is to A B - - - - - - - - - - - - 2.2929272
So is sin ∠ B - - - 46° 10′ - - 9.8581505

To height A C - - 141.61 - - 2.1510777

3. It is required to find the perpendicular height of a cloud, or other object, when its angles of elevation, as taken by two observers at the same time, on the same side of it, and in the same vertical plane, were 64° 10′ and 35° 25′; their distance asunder being half a mile, or 880 yards.



∠ A B C - - - - - 64° 10′
∠ A D C - - - - - 35° 25′
∠ D A B - - - - - 28° 45′

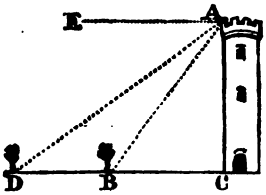
As sin ∠ D A B - - 28° 45′ - - - 9.6821349
0.3178651
Is to opp. side D B - 880 - - - - 2.9444827
So is sin ∠ A D C - - 35° 25′ - - 9.7630671

To opp. side A B - - 1060.29 - - 3.0254149

Then,

As rad, or sin ∠ C	90°	10.0000000
Is to opp. side A B	1060.291	3.0254149
So is sin ∠ A B C	64°. 10′	9.9542741
To height A C	954.31 yds.	2.9796890

4. From the top of a tower, 125 feet high, which lay in the same right line with two trees, I took the angles of depression formed by lines conceived to be drawn from my eye to the bottom of each tree, and another line parallel to the horizon, and found them to be 56°¼ and 40½: what is the distance of the two objects (*r*)?



As rad, or sin	90°	10.0000000
Is to A C	125 feet	2.0969100
So is cot ∠ A B C	56° 15′	9.8248926
To B C	83.522	1.9218026

Then,

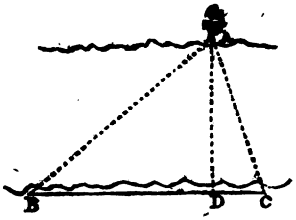
As rad, or sin	90°	10.0000000
Is to A C	125	2.0969100
So is cot ∠ A D C	40° 30′	10.0685011
To D C	146.351	2.1654111

83.522

Dist. D B 62.829

(*r*) An angle taken from the top of any object, or the one usually called the angle of depression, is that which is made by a right line

6. Wanting to know the breadth of a river, I measured a distance of 120 yards in a straight line by the side of it, and at each end of this line, I found the angles subtended by the other end and a tree, close by the opposite side of the river, to be 51° 45′ and 78 50 : what is its perpendicular breadth?



∠ B - - - - 51° 45′
∠ C - - - - 78° 50′
130° 35′
180° 0′
∠ B A C - - - 49° 25′

As sin ∠ B A C - 49° 25′ - - - 9.8805052
0.1194948
Is to B C - - - - 120 - - - - - - 2.0791812
So is sin ∠ C - - 78° 50′ - - - 9.9916991
To A B - - - - - - - - - - - - - - - - 2.1903751

Then,

As sin ∠ D - - - 90° - - - - - 10.0000000
Is to A B - - - - - - - - - - - - - 2.1903751
So is sin ∠ B - - 51° 45′ - - - - 9.8950450
To breadth A D - 121.74 - - - 2.0854201

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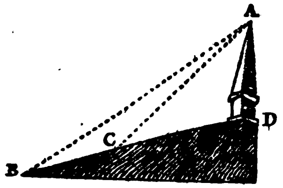
passing from the eye to the object and another line drawn parallel to the horizontal plane. Thus, E A B is the ∠ of depression of the object B, which by the nature of parallel lines is equal to the ∠ A B C; and its complement is the ∠ B A C.

6. Wanting to find the height of an obelisk, standing on the top of a declivity, I measured from its bottom a distance of 60 yards, and there found the angle formed by the plane and an imaginary line drawn to the top of the object to be 45½°; and after measuring on, in the same direction, 40 yards further, the angle, formed as before, was only 24° 45′: what was the height of the obelisk?



∠ACD - - 45° 30′
∠ABC - - 24° 45′
∠BAC - - 20° 45′

Then, in the triangle BAC,

As sin ∠BAC - - -	20° 45′ - -	9.5493602
		0.4506398
Is to opp. side BC -	40 yards -	1.6020600
So is sin ∠ABC - -	24° 45′ - -	9.6218612
To opp. side AC - -	47.268 yards	1.6745610

In the triangle ACD,

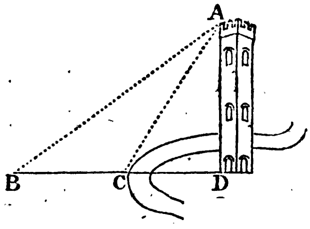
As sum sides CA, CD	107.268 - -	2.0304378
		7.9695622
Is to diff. sides CA, CD	12.732 - -	1.1048966
So is tan ½ sum ∠ˢA, D	67° 15′ - -	10.3774391
To tan ½ diff. ∠ˢA, D	15° 48′ - -	9.4518979

And 67° 15′ − 15° 48′ = 51° 27′ ∠CAD.

Lastly, in the same triangle A C D,

As sin ∠ C A D - -	51° 27′ - -	9.8932426
		0.1067574
Is to opp. side C D -	60 yards - -	1.7781513
So is sin ∠ C - - -	45° 30′ - -	9.8532421
To height A D -	54.721 yards, or 164.163 feet	1.7381508

7. Wanting to know the height of an inaccessible object, I took its angle of elevation, at the least distance I could from its bottom, which was found to be 57¾°; and going 105 yards further, in a right line, the angle was then found to be only 32½°; required its height, and my distance from it at the first station, the instrument being 5 feet above the ground at each observation.



∠ A C D - - -	57° 45′
∠ A B C - - -	32° 30′
∠ B A C - - -	25° 15′

Then in the triangle A B C,

As sin ∠ B A C - - -	25° 15′ - -	9.6299890
		0.3700110
Is to opp. side B C -	105 - - - -	2.0211893
So is sin ∠ B - - -	32° 30′ - -	9.7302165
To opp. side A C - - - - - - - - - -		2.1214168

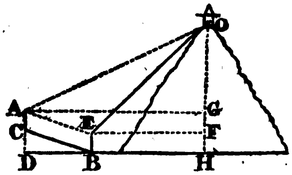
And, in triangle ACD,

As sin ∠ D - - - -	90° - - - -	10.0000000
Is to opp. side AC - - - - - - - -		2.1214168
So is sin ∠ C - - -	57° 45′ - -	9.9272306
To opp. side AD - -	101.85 - -	2.0486474
	1.66	
	100.19 yds. height AD.	

Lastly, in the same triangle ACD,

As sin ∠ D - - - -	90° - - - -	10.0000000
Is to opp. side AC - - - - - - - -		2.1214168
So is cos ∠ C - - -	57° 45′ - -	9.7272276
To side CD - - - -	70.574 yds.	1.8486444

8. Wanting to know the height and distance of the object O, on the top of a hill, I measured from the station B, a base BC of 130 yards, up sloping ground, directly from O; and having set up a staff BE, of an equal height to that of the theodolite AC, I found the angle of elevation OAG, at the station C, to be 27° 10′, and the angle of depression GAE, of the top of the staff E, 8° 45′; and at the station B the angle of elevation OEF, was 39° 40′: required the horizontal distance BH, the height OH, and the height CD of the station C above B.



Here, according to the question, we have the following angles:

∠ OAG = 27° 10′
∠ GAE = 8° 45′
∠ OAE = 35° 55′
131° 35′
35° 55′
167° 30′

∠ AEF = 171° 15′ sup^t ∠ GAE
∠ OEF = 39° 40′
∠ OEA = 131° 35′
180° 0′
167° 30′
12° 30′ ∠ AOE

Then,

: Sin ∠ AOE - - 12° 30′ - - 9.3353368
0.6646632

: AE - - - - - - 130 - - - - 2.1139434
: : Sin ∠ OAE - - 35° 55′ - - 9.7683480
: OE - - - - - - - - - - - - - 2.5469546

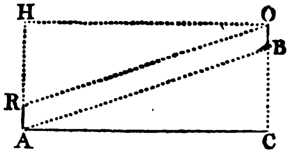
: Rad, or sin ∠ EFO 90° - - - - 10.0000000
: OE - - - - - - - - - - - - - - - 2.5469546
: : Sin ∠ OEF - - 39° 40′ - - 9.8050385
: OF - - - - - - - - 224.91 - - 2.3519931

: Rad, or sin - - - 90° - - - - 10.0000000
: OE - - - - - - - - - - - - - - - 2.5469546
: : Cos ∠ OEF - - - 39° 40′ - - 9.8863616
: EF, or BH - - - 271.21 - - 2.4333162

: Rad, or sin - - - 90° - - - - 10.0000000
: AE or CB - - - - 130 - - - - 2.1139434
: : Sin ∠ CBD or GAE 8° 45′ - - 9.1821960
: CD - - - - - - - - 19.776 - - 1.2961394

And if EF, or FH, the height of the theodolite, be added to OF, it will give the height of O above the horizontal line DH.

It may here be further remarked, that in certain trigonometrical operations, when a base is measured on sloping ground, it is sometimes necessary to reduce it to the corresponding horizontal line; which may be done thus.

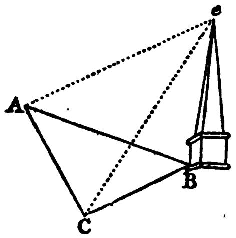


Let A B be the measured base, O B a theodolite, and A R a staff equal in height to the instrument: also, suppose H O R to be the angle of depression of the top R of the staff, below the horizontal line H O; then, if C O be perpendicular to H O, the line A C, which is parallel to H O, will be the horizontal base corresponding to A B. And, by case I. of right-angled triangles:

As rad : A B :: cos H O R (or B A C) : A C (s).

$$\text{Or, } AC = \frac{AB \times \cos HOR}{\text{rad.}}.$$

It also frequently happens that the angles subtended by distant objects, lie in planes oblique to the horizon, in which case they may be reduced to the corresponding horizontal angles, as follows:



(s) If A B be 300 yards, and the ∠ of depression H O R 5°, the horizontal line A C will be 298.9 yards, differing from the mea-

Let A C be any two points in the horizontal plane A B C, e B a distant spire, or other object, e A C, e C A, the oblique angles, taken at A and C, and e A B, e C B, the angles of elevation.

Then,

Cos $\angle$ elevation e A B : cos given $\angle$ e A C :: rad. or sin 90° : cos reduced $\angle$ B A C.

And cos $\angle$ elevation e C B : cos given $\angle$ e C A :: rad. or sin 90° : cos reduced $\angle$ A C B (*t*).

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sured base A B by only $1\frac{1}{10}$ yards; so that, except in cases where great accuracy is required, a reduction of this kind seems unnecessary, when the measured base is inclined to the horizon in a small angle.

(*t*) For by right $\angle^d$ Δ^s, $eB^2 = eA^2 - AB^2 = eC^2 - CB^2$; hence, also, $eA^2 - eC^2 = AB^2 - CB^2$; and by adding AC^2 to both sides of this equation, and then dividing each of them by 2 A C, we shall have $\frac{eA^2 + AC^2 - eC^2}{2AC} = \frac{AB^2 + AC^2 - CB^2}{2AC}$.

But by the demonstration of Theorem IV. in Plane Trigonometry, Part 2, it is shown that $\cos BAC = r\left(\frac{AB^2 + AC^2 - CB^2}{2AC \times AB}\right)$, and $\cos eAC = r\left(\frac{eA^2 + AC^2 - eC^2}{2AC \times eA}\right)$.

Hence $AB \cos BAC = eA \cos eAC$; or $\cos BAC = \frac{eA}{AB} \cos eAC$.

But, by right $\angle^d$ Δ^s, $eA : AB :: r : \cos eAB$; and consequently $\frac{eA}{AB} = \frac{r}{\cos eAB}$.

Whence, by substitution, we have $\cos BAC = \frac{r \cos eAC}{\cos eAB}$.

And in the same manner it may be shown that $\cos ACB = \frac{r \cos eCA}{\cos eCB}$.

Also $\angle$ A*e*C will be reduced to $\angle$ ABC, by taking $\angle$ BAC + $\angle$ ACB from 180°. And if AC be a known base, the horizontal distances AB, CB, may be determined by case I. of oblique-angled triangles.

But if it should be required to reduce the angle A*e*C, taken at the top of any eminence *e*B, to its corresponding horizontal angle ABC, by employing only the observed angle and those of depression, it may be done by the following rule: $\sin \frac{1}{2}\, ABC =$

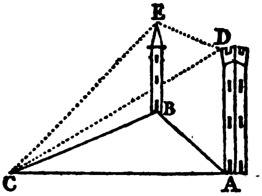
$$r\sqrt{\frac{\sin \frac{1}{2}(AeC + eAB - eCB)\,\sin \frac{1}{2}(AeC + eCB - eAB)}{\cos eAB \cos eCB}}.$$

Where *e*AB, *e*CB, are the angles of depression of the two distant objects A, C, A*e*C the observed angle, and $r =$ rad, or sine 90° (*u*).

(*u*) In the Δ ABC, we have, by the demonstration of the last problem, $\cos ABC = r\left(\frac{AB^2 + BC^2 - AC^2}{2AB \times BC}\right)$, or, $AC^2 = AB^2 + BC^2 - \frac{2AB \times BC \cos ABC}{r}$; and in Δ A*e*C, $\cos AeC = r\left(\frac{eA^2 + eC^2 - AC^2}{2eA \times eC}\right)$, or $AC^2 = eA^2 + eC^2 - \frac{2eA \times eC \cos AeC}{r}$. Whence, by equality and transposition, $eA^2 - AB^2 + eC^2 - BC^2 = \frac{2eA \times eC \cos AeC}{r} - \frac{2AB \times BC \cdot \cos ABC}{r}$, or, (because both $eA^2 - AB^2$, and $eC^2 - BC^2 = eB^2$) it will be $eB^2 = \frac{eA \times eC \cos AeC}{r} - \frac{AB \times BC \cos ABC}{r}$.

But, by Plane Trigonometry, $eB = \frac{eA \sin eAB}{r}$, or, $\frac{eC \sin eCB}{r}$, $AB = \frac{eA \cos eAB}{r}$, and $BC = \frac{eC \cos eCB}{r}$; whence, by substitution, $\frac{eA \times eC \sin eAB \sin eCB}{r^2} = \frac{eA \times eC \cos AeC}{r} - \frac{eA \times eC \cos eAB \cos eCB \cos ABC}{r^3}$, or, $r \sin eAB \sin eCB$

The same formula may also be applied to the reducing an angle E C D, subtended by any two distant objects E B, D A, when taken at a point C in the horizontal plane A B C, by using only the observed angle and those of elevation.



$= r^2 \cos \mathrm{A}\,e\,\mathrm{C} - \cos e\,\mathrm{A\,B} \cos e\,\mathrm{C\,B} \cos \mathrm{A\,B\,C}$; and consequently,
$\cos \mathrm{A\,B\,C} = r \left(\frac{r \cos \mathrm{A}\,e\,\mathrm{C} - \sin e\,\mathrm{A\,B} \sin e\,\mathrm{C\,B}}{\cos e\,\mathrm{A\,B} \cos e\,\mathrm{C\,B}} \right)$.

But since, by Article 26, Part 2, $\cos \mathrm{A\,B\,C} = \frac{r^2 - 2 \sin^2 \frac{1}{2} \mathrm{A\,B\,C}}{r}$,

we shall have $\sin^2 \frac{1}{2} \mathrm{A\,B\,C} = \frac{r}{2} (r - \cos \mathrm{A\,B\,C}) = \frac{r}{2}$

$$\times \left(r - \frac{r^2 \cos \mathrm{A}\,e\,\mathrm{C} - r \sin e\,\mathrm{A\,B} \sin e\,\mathrm{C\,B}}{\cos e\,\mathrm{A\,B} \cos e\,\mathrm{C\,B}}\right) = \frac{r}{2}$$

$$\times \left(\frac{r \cos e\,\mathrm{A\,B} \cos e\,\mathrm{C\,B} + r \sin e\,\mathrm{A\,B} \sin e\,\mathrm{C\,B} - r^2 \cos \mathrm{A\,B\,C}}{\cos e\,\mathrm{A\,B} \cos e\,\mathrm{C\,B}}\right)$$

$$= \text{(by Article 20) } \frac{r}{2} \left\{ \frac{r^2 \cos (e\,\mathrm{A\,B} - e\,\mathrm{C\,B}) - r^2 \cos \mathrm{A}\,e\,\mathrm{C}}{\cos e\,\mathrm{A\,B} \cos e\,\mathrm{C\,B}} \right\}$$

$$= \frac{r^3}{2} \left\{ \frac{\cos (e\,\mathrm{A\,B} - e\,\mathrm{C\,B}) - \cos \mathrm{A}\,e\,\mathrm{C}}{\cos e\,\mathrm{A\,B} \cos e\,\mathrm{C\,B}} \right\} = \text{(by Article}$$

$$\text{29)} \ \frac{r^2 \sin \frac{1}{2} (e\,\mathrm{A\,C} + e\,\mathrm{A\,B} - e\,\mathrm{C\,B}) \sin \frac{1}{2} (\mathrm{A}\,e\,\mathrm{C} + e\,\mathrm{C\,B} - e\,\mathrm{A\,B})}{\cos e\,\mathrm{A\,B} \cos e\,\mathrm{C\,B}};$$

Whence, $\sin \frac{1}{2} \mathrm{A\,B\,C}$

$$= r \sqrt{\frac{\sin \frac{1}{2} (e\,\mathrm{A\,C} + e\,\mathrm{A\,B} - e\,\mathrm{C\,B}) \sin \frac{1}{2} (\mathrm{A}\,e\,\mathrm{C} + e\,\mathrm{C\,B} - e\,\mathrm{A\,B})}{\cos e\,\mathrm{A\,B} \cos e\,\mathrm{C\,B}}}$$

as by the rule.

And in the same manner it may be shown for the following formula, that $\cos \mathrm{B\,C\,A} = r \left(\frac{r \cos \mathrm{E\,C\,D} - \sin \mathrm{E\,C\,B} \sin \mathrm{D\,C\,A}}{\cos \mathrm{E\,C\,B} \cos \mathrm{D\,C\,A}} \right)$
And the $\sin \frac{1}{2} \mathrm{B\,C\,A}$ as it is there given.

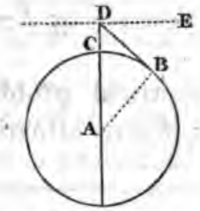
Thus, $\sin \frac{1}{2}\,\mathrm{BCA} =$

$$r\sqrt{\frac{\sin\frac{1}{2}(\mathrm{ECD}+\mathrm{ECB}-\mathrm{DCA})\sin\frac{1}{2}(\mathrm{ECD}+\mathrm{DCA}-\mathrm{ECB})}{\cos\mathrm{ECB}\cos\mathrm{DCA}}}.$$

Where E C B, D C A, are the angles of elevation of the two objects, E B, D A, and E C D the observed angle.

Besides this reduction of angles to the plane of the horizon, it is also sometimes necessary to attend to what is usually called the *depression* or *dip* of the horizon, which makes the observed angle of elevation greater than it would otherwise be.

Thus, if an observer, whose eye is at D, takes the altitude of an object with the sextant, by bringing it to the water's edge at B, instead of to the horizon D E, the altitude will, evidently, be too great by the angle E D B, which angle may be found by the following rule:

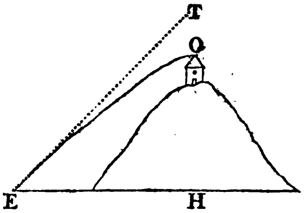


$$\mathrm{Cos}\ \angle\,\mathrm{EDB} = \frac{\mathrm{rad}\times\mathrm{AB}}{\mathrm{AC}\ (\text{or}\ \mathrm{AB})+\mathrm{DC}} = \frac{\mathrm{rad}\times\mathrm{AB}}{\mathrm{AD}}.$$

Where A B or A C is the radius of the earth, which is known to be 3979 miles, and D C the height of the observer's eye above the surface of the earth.

Another source of error, in taking angles of elevation, arises from the effect of *refraction*, which always makes objects appear more elevated than they really are, on account of the rays of light, in their passage through the atmosphere, being continually bent downwards, and coming to the eye in the form of a curve, instead of proceeding in a right line.

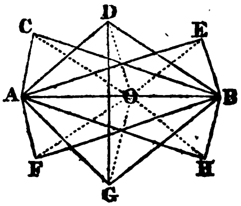
Thus, if E be the place of the observer's eye, E H the horizontal plane, and O any elevated distant object, this object will not appear in its true place at O, but will be seen in the direction E T, which is a tangent to the curve at the point E; and therefore the apparent angle of elevation T E H will be greater than the true angle of elevation O E H, by the angle T E O, considering E O as a right line (v).



Various trigonometrical problems may also be formed from the different situations which objects may be supposed to have with respect to each other; but, in general, these are only applications of the preceding rules.

(v) This kind of refraction, which is called terrestrial, in order to distinguish it from that which affects the altitudes of the heavenly bodies, is not constant at the same elevation and distance; but is found to vary with the density and other changes in the atmosphere. At the distance of 8 or 10 miles, it is sometimes no more than about 30″; but, in particular states of the air, it has been found to amount to upwards of 2′. Maskelyne makes it $\frac{1}{10}$ of the intermediate arc C B (See fig. to dip of horizon) between the observer and the object: Bouguer $\frac{1}{7}$, Legendre $\frac{1}{14}$, and Gen. Roy from $\frac{1}{5}$ to $\frac{1}{24}$; which allowances differ too much from each other for any reliance to be placed on them.

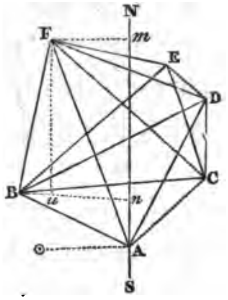
As, for instance, the distances of the most remarkable places in a town, or of several villages from each other, the plan of a camp, or of a country, &c. may be taken from what has been already explained.



Thus, if A, C, D, E, B, H, G, F, &c. be several objects, the situations of which are to be laid down in a map, or plan; choose a convenient situation A B for a base, from which you can see all the objects, and let it be as long as possible, in proportion to the most distant of them. Then, from the extremity A, measure the angles E A B, D A B, C A B, &c. H A B, G A B, F A B, &c. And from the other extremity B measure the angles C B A, D B A, E B A, &c. F B A, G B A, H B A, &c. And as the common base A B, and the several angles of all the triangles are now known, the sides A C, A D, A E, &c. and consequently the points C, D, E, &c. may be determined by the first rule in plane trigonometry.

But, in order to insure the accuracy of the operation, the objects C, D, E, &c. should be all intersected from some third station O, in the base A B; or otherwise the figure may appear, in the plotting of it, to be right, when it is not so, and there will be no means of knowing whether the angles have been justly taken, without going over the work again.

A measurement may also be carried on, or the distance of any two remote places may be found, by means of a series of triangles, formed from a measured base, in a manner similar to that generally practised in taking the trigonometrical survey of a country.



Thus, let A B be the measured base, and C, D, any two objects that can be seen from the stations A, B; then if the angles C A B, C B A, D A B, D B A, be taken with a theodolite, or other instrument, we can thence find the sides C B, C A, D B, D A; and the angle B D A.

Also, knowing the angles D A B, C A B, we know their difference D A C; from which, and the two sides C A, D A, we can find the side C D, and the angles A D C, A C D. And if E, F, be any other two objects, visible from C D, we can determine the lengths of the sides D E, D F, and the angles at E and D in a similar manner.

And in the same way the operations may be continued from one base to another to any distance; but, in order to render the conclusion more accurate, the mensuration should be carried on from one base to the next by different sets of triangles, all leading to the same two objects, and then taking the mean of the results.

The distances from the first stations to the last, may also be readily determined; for the two sides D F, D A, and the included angle F D A being known, we can thence find the side A F; and from the sides D B, D F, and the included angle B D F, we can find the side B F.

It will be proper, however, in examples of this kind, to find every angle of the triangles by observation, if the situations will permit, as the difference of their sum from 180° will enable us, in some measure, to judge of the accuracy of the work. All the principal distances should also be laid down from a scale of equal parts; because a triangle can be more accurately constructed by means of its sides than from the angles (*w*).

(*w*) After carrying on a series of triangles, in the manner here described, to some distance, it is customary to find, by an actual measurement, the interval of two objects whose distance has been previously obtained by calculation, in order to determine the error of the calculated distance: which line, so measured, is called the *Base of Verification*.

A survey of this kind is at present carrying on in this country, from a base of 27406½ feet, first measured on *Hounslow Heath*; from which it appears, that by continuing the measurement to *Salisbury Plain*, through a tract of near 60 miles, the distance of two objects was there found, by calculation, from the mean result of several series of triangles, to be 36574.4 feet; and, by

When this part of the work is finished, the distance A F may be reduced to the meridian N S; and the length of the corresponding arc A *m*, of that circle, be determined, as follows:

Let O be the point of the horizon where the sun sets, and take the angle O A B; then supposing the latitude of the place A to be known, the sun's amplitude O A N can be readily found, and consequently the difference of these angles B A N.

Hence, drawing B *n*, F *m* perpendicular to the meridian, we have B A, A *n*, and the angle B A *n* given; from which the angle A B *n*, and the sides A *n* and B *n* may be found; also, if from the angle A B C there be taken the angle A B *n*, the remainder, added to C B F, will give the angle F B *u*; from which, and the side F B, we can find F *u*, or its equal *m n*; and thence the whole meridional distance, or length of the arc A *m*.

This being done, the difference of latitude between the two stations A and F may also be easily determined; for by taking, with a proper instrument, the true zenith distances of a star, at each of these places, their

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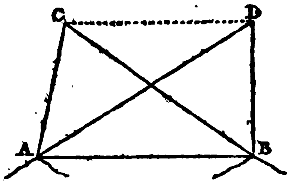
an actual measurement, it was found to be 36574.3, differing but little more than an inch from the computed distance.

The area, or content, of any extent of land, measured in this way, may be found thus:—Area of the $\triangle ABC = \frac{1}{2} AB \times BC \times \sin \angle ABC$, radius being unity; area of $BCD = \frac{1}{2} CB \times BD \times \sin \angle CBD$; area of $BDE = \frac{1}{2} BD \times BE \times \sin \angle EBD$; area of $EBF = \frac{1}{2} EB \times BF \times \sin \angle EBF$; and thus we may proceed for any number of triangles into which the whole is divided.

difference will give the difference of latitude sought; and by proceeding in a similar way, the same may likewise be determined between A and either of the other stations (*x*).

It also sometimes happens, in making a survey, that the distance between two objects C, D, having been determined, it is required to find the distance A B of two eminences A, B, which are conveniently situated for extending the series of triangles.



This is done by measuring the angles C A D, C A B, D B C, D B A: and as there are not sufficient data in any of the triangles to compute the other parts, we must assume a value for A B, and thence compute the value of C D, as in the last proposition. Then, as the computed value of C D is to its true value, so is the assumed value of A B to its true value (*y*).

Military sketches, or small surveys, where much accuracy is not required, may also be taken by means

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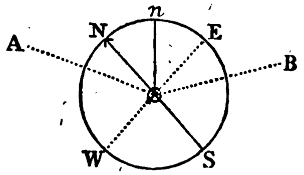
(*x*) For a more ample detail of particulars respecting this part of the subject, see *Trigonométrie de Cagnoli*, 2d Edit. and the *Traité de Géodésie* and *Traité de Topographie de Puissant*.

(*y*) For, by changing the value of A B while the angles at A and B remain the same, the whole figure will continue similar to itself; and consequently A B will vary in the same proportion as C D.

of a pocket compass, fitted to the top of a staff, which being stuck in the ground, so that the needle may play freely, the angular distances, or bearings, must then be taken from the magnetic meridian.



Thus, let N S represent the needle, or magnetic meridian, *n* the true north point, and E, W, the east and west points; then, if the sights of the compass be directed to the object A, and the angle N O A, for example, be 40°, the object is said to bear N. W. 40°; and if the sights, when directed to the object B, make the angle N O B 110°, it is said to bear N. E. 110°.

The compass will, likewise, be found useful in reconnoitring a country with a map or plan, when the direction of the meridian is laid down, and we know the magnetic variation; and, in such cases, a distance may be measured by pacing, in order to adapt a scale to the plan or sketch.

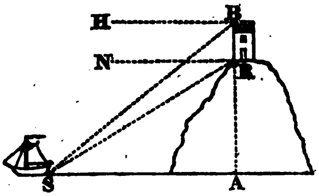
Note. The variation of the magnetic needle is, at this time, between 23° and 24° westward, at London.

MISCELLANEOUS EXAMPLES.

1. At B, the top of a castle, which stood on a hill near the sea shore, the ∠ of depression H B S, of a ship at anchor, was 5° 4′, and at R, the bottom of the castle, its depression N R S was 4° 46′, required the ho-

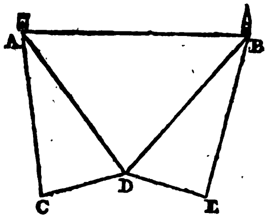
rizontal distance of the vessel, and the height of the building above the level of the sea, supposing the castle itself to be 68 feet high.

Ans. AS 12891.61 feet, and AB 1142.986 feet.



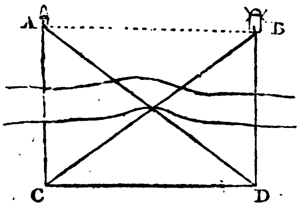
2. Wanting to know the distance between two objects A, B, which could only be seen from a particular place D, I set up two poles at C, E, and took the ∠s ADC 89° 10′, ADB 72° 35′, and BDE 54° 36′. I then measured a distance DE of 216 yards, and another DC of 200, and took the ∠s BED 88° 34′, and DCA 50° 27′: required the distance AB.

Ans. AB 367.56 yards.



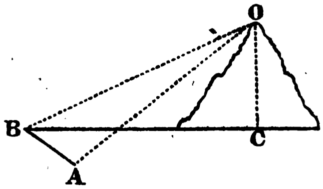
3. Wanting to know the distance between two inaccessible objects A, B, I measured a base CD of 500 yards: at C the ∠ BCD was 48° 20′, and the ∠ ACB 47° 48′; and at D the ∠ CDA was 54° 19′ and ∠ ADB 47° 15′; required the distance AB.

Ans. AB 742.682 yards.



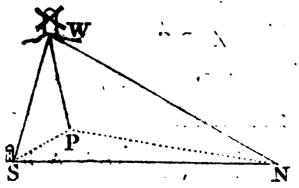
4. Wanting to know the distance A C of a hill from the station A, and also its height O C, we measured a base A B of 376 yards, on ground nearly level, and at the extremities A, B, observed the horizontal angles B A O 43° 17′, and A B O 78° 29′, and at A, the angle of elevation O A C was 4° 41′: required the distance A C, and height C O.

Ans. A C 431.898, O C 35.382 yards.



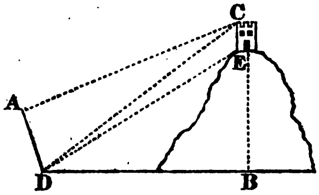
5. At a mile-stone N on the ascending road N S, I observed the angle S N W between the next mile-stone S and the windmill W, on the top of a hill, and found it to be 47° 36′, and the angle of elevation W N P was 3° 47′; also at the mile-stone S, the angle N S W was 91° 5′: from which it is required to find the horizontal distance N P, and the height P W.

Ans. N P 2659.492, and P W 175.8662 yds.



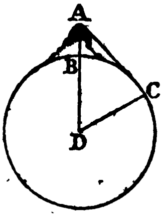
6. Wanting to know the height of a castle C E, standing upon a hill, and the ground not permitting me to retreat from it in a right line, I measured a base D A of 57 yards, and at D took the angles C D B 59°, C D E 26°, and A D C 72° 20′. At A I also took the angle C A D 65° 30′: from which it is required to find the height of the castle C E, and that of the hill B E, above the level of the first station D.

Ans. C E 40.38671, B E 25.8431 yards.



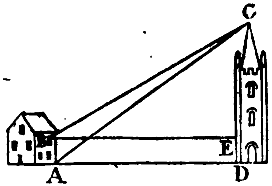
7. It is required to find how far the Peak of Teneriffe can be seen at sea, supposing its height A B to be 2 miles, and the radius of the earth D C 3979 miles.

Ans. dist. A C 126.2036 miles.

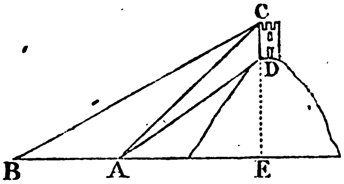


8. From a window A, near the bottom of a house, which seemed to be on a level with the bottom of a church C D, I took the ∠ of elevation C A D of the top of the steeple equal to 40° 21′, and from another window B, 20 feet directly above the former, the ∠ of elevation C B E was 37° 36′: from which it is required to find the height and distance of the steeple.

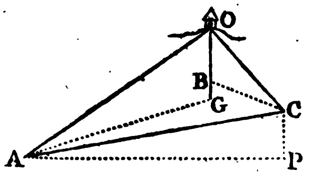
Ans. C D 213.8357, A D 251.7008 feet.



9. Being at the station A, on an horizontal plane, and wanting to know the height of a tower C D, placed on the top of an inaccessible hill, I took the angle of elevation D A E, of the top of the hill, equal to 40° 15′, and of the top of the tower C A E equal to 51° 13′; then, measuring on, in a direct line from it, to the distance A B of 110 yards, I found the ∠ of elevation of the top of the tower C B E to be 34° 45′: what then is its height? Ans. height C D 55.1337 yds.



10. Suppose A and C to be two stations on sloping ground, O an object on the top of a hill, and the ∠s O C A, O A C, measured with a sextant, to be 78° 23′ and 64° 11′ respectively: also, suppose the ∠ of elevation at A is 7° 36′, and at C 6° 22′; what are the horizontal distances and height of the object, A C being 430 yards? Ans. A G 686.848, C B 632.891, O B 70.61731, O G 91.64509

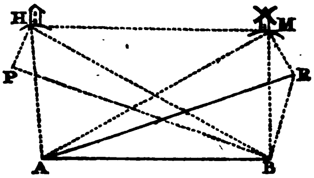


11. Wanting to know the distance between two inaccessible objects H, M, and also their heights M R, H P, we measured a base A B of 680 yards, on ground nearly horizontal, and at the extremities A, B, took the following angles:

At A ∠ { BAM 40° 26′ / MAH 57° 30′ }
At B ∠ { ABH 42° 26′ / HBM 72° 7′ }
of Alt. { MAR 7° 58′ / HBP 3° 47′ }

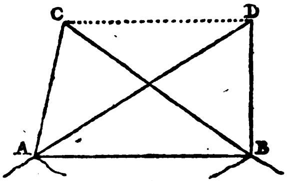
From which it is required to find the heights and distance.

Ans. HM 1235.38 yds. MR 202.7185, and HP 69.668



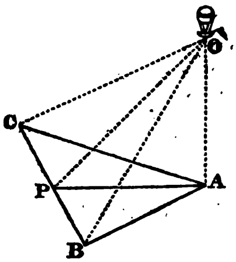
12. Knowing the distance of two inaccessible objects C, D, to be 2860 yards, and wishing to determine the distance of two other objects A, B, I took the angles C A D 49° 20′, C A B 88° 39′, D B C 53° 16′ and D B A 84° 39′; from which it is required, by means of an assumed distance, to find the true distance A B.

Ans. A B 3080.49 yards.



13. In the year 1784, a base B C being measured on Blackheath, of a mile in length, the angles of eleva-

tion of Lunardi's balloon were taken, at the same time, by observers placed at its two extremities and in the middle; the one at B being 46° 10′, that at P 55° 8′, and that at C 54° 30′: required the height O A of the balloon (y). Ans. O A 1.04298 miles.



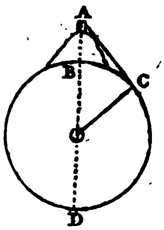
14. Supposing it were possible to see a light-house or other object A, in the horizon, at the distance of 180 miles, it is required to find its height A B, the circumference of the earth B C D being 25000 miles.

Ans. $4\frac{1}{10}$ miles nearly, being about the height of

(y) As the height O A, in this question, cannot be determined by any of the direct rules before given, it will be proper to show how the formula for the solution of it is obtained; which may be done thus: By case IV, of right ∠d Δs, $AB = \frac{OA \cot OBA}{r}$, $AP = \frac{OA \cot OPA}{r}$, $AC = \frac{OA \cot OCA}{r}$: And by a well-known prop. in Geometry, $AB^2 + AC^2 = 2AP^2 + 2BP^2$, or $AB^2 + AC^2 - 2AP^2 = 2BP^2$; hence, by substitution, $\frac{OA^2 \cot^2 OBA}{r^2} + \frac{OA^2 \cot^2 OCA}{r^2} - \frac{2OA^2 \cot^2 OPA}{r^2} = 2BP^2$, or $OA^2 = \frac{2r^2 BP^2}{\cot^2 OBA + \cot^2 OCA - 2\cot^2 OPA}$; and consequently

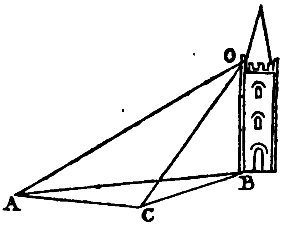
$$OA = \sqrt{\frac{2r^2 BP^2}{\cot^2 OBA + \cot^2 OCA - 2\cot^2 OPA}}$$

Chimborazo, the highest mountain of the Andes, and, probably, in the world.



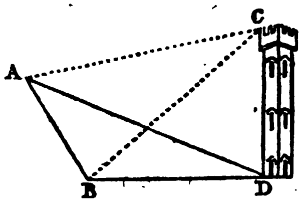
15. Suppose O B to be an object standing on the horizontal plane A B C, and that A C is a measured base of 260 yards, at the extremities of which A, C, the following angles have been taken, O A C 54° 46′, O C A 63° 54′, and the ∠ of elevation O A B 6° 42′; it is required to find its height O B, the reduced ∠ A B C, and the horizontal distances A B, C B.

Ans. A B 264.2875, C B 240.0387, O B 31.0466 yds. and ∠ A B C 61° 50′.



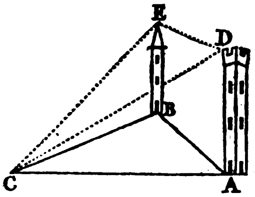
16. At the top of a tower C D 160 feet high, I took the angle A C B, subtended by two distant objects A, B, 69° 37′, also the angles of depression C A D 1° 46′, and C B D 2° 17′, from which it is required to find the reduced ∠ A D B, and the distances A B, B D, D A.

Ans. ∠ A D B 65° 43′, B D 4012.76, D A 5187.408, A B 5342.652.



17. At the point C in the horizontal plane A B C, I took the angle E C D 58° 20′ subtended by the tops of two towers, whose heights D A, E B, were known to be 169 and 213 feet respectively, and also the angles of elevation D C A 3° 18′, and E C B 4° 21′; from which it is required to find the distances C A, C B, and A B.

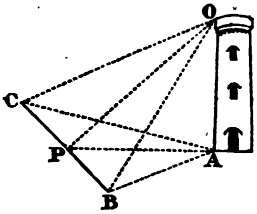
Ans. C A 2930.993, C B 2800.125, A B 2801.2579.



18. Observing an object O A at a distance, I took its angle of elevation O B A at the place where I stood, and found it to be 50° 23′; I then measured a distance P B of 60 yards, in the most convenient direction the ground afforded, and at this station found its elevation O P A to be 40° 33′; after which I measured on, in the same line, 50 yards further to C, and at this place found its elevation O C A to be 30° 40′; from which it is required to determine the height of the object, and its distance from each of the three stations B, P, C (z).

Ans. A B 65.417, A P 92.368, A C 132.573, and height O A 79.029.

(z) The height O A, in this question, may be obtained in a similar manner with that of the balloon, in quest. 14, as follows:



19. Observing three objects A, B, C, whose distances asunder are known to be as follows: AB 5307 yards, BC 7022 yards, and AC 3556 yards, I took their angles of position from the place where I stood D, which was furthest from the object A, and found the

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By case IV, right ∠ᵈ Δˢ, $AB = \frac{OA \cot OBA}{r}$, $AP = \frac{OA \cot OPA}{r}$, and $AC = \frac{OA \cot OCA}{r}$. But, by case VI, of oblique ∠ᵈ Δˢ, $\cos BPA = r \times \frac{BP^2 + AP^2 - AB^2}{2BP \times PA}$, and $\cos CPA$, or its equal $-\cos BPA = -r \times \frac{CP^2 + AP^2 - AC^2}{2CP \times PA}$. Hence, $\frac{BP^2 + AB^2 - AB^2}{BP} = -\frac{CP^2 + AP^2 - AC^2}{CP}$; or, by transposition and addition, $CP \times AB^2 + BP \times AC^2 - (CP + BP) \times AP^2 = (CP + BP) \times CP \times BP$. And consequently, by substitution, $CP \times \frac{OA^2 \cot^2 OBA}{r^2} + BP \times \frac{OA^2 \cot^2 OCA}{r^2} - (CP + BP) \times \frac{OA^2 \cot^2 OPA}{r^2} = (CP + BP) \times CP \times BP$; or, by reducing and simplifying the equation

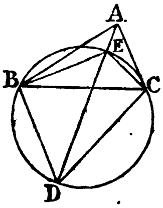
$$OA^2 = \frac{r^2 (CP + BP) \times CP \times BP}{CP \cot^2 OBA + BP \cot^2 OCA - (CP + BP) \cot^2 OPA};$$

$$\text{or, } OA = r \sqrt{\frac{(CP + BP) \times CP \times BP}{CP \cot^2 OBA + BP \cot^2 OCA - (CP + BP) \cot^2 OPA}};$$

by which formula the height OA may be readily computed.

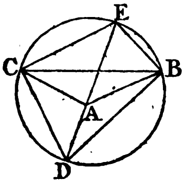
∠ A D C to be 17° 35′, and B D A 26° 21′; required my distance from each of the objects.

Ans. D B 8544.854, D A 11369.175, D C 9916.558 yds.

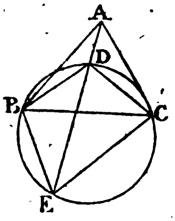


20. Supposing the objects A, B, C, as seen from the point D, which is the nearest to A, to stand as in the figure below; and that their distances are A B 5307, B C 7022, and A C 3556 yards, the angle B D A being 27° 13′, and C D A 29° 25′; it is required to determine the distances D A, D B, D C.

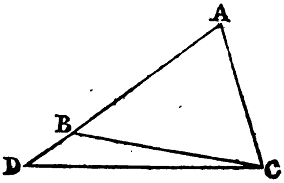
Ans. D A 3494.058, D C 6158.059, D B 8167.893 yards.



21. Supposing the three objects A, B, C, as seen from a point D, within the triangle, to stand as below; and that their distances are A B 5307, B C 7022, and A C 3556 yards, the angle B D C being 122° 37′, and A D C 131° 54′; required the distances D A, D C, and D B. Ans. D A 870.925, D B 5007.778, D C 2914.783 yards.

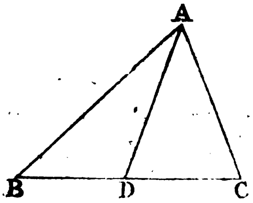


22. Supposing the objects A, B, C, as seen from a point D, in the line A B produced, to stand as in the figure below; and that their distances are A B 5307, B C 7022, and A C 3556 yards, the angle B D C being 18° 7′; it is required to find the distances D A, D C, D B. Ans. D B 4481.997, D C 11142.11, D A 9788.997 yds.



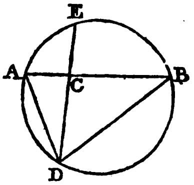
23. Supposing the three objects A, B, C; as seen from a point D, in the base B C, to stand as below; and that A B is 5307, A C 3556, and B C 7022 yards, the angle A D B being 108° 5′: required the distances D A, D C, and D B.

Ans. D B 3761.025, D A 2754.544, and D C 3260.975 yards.



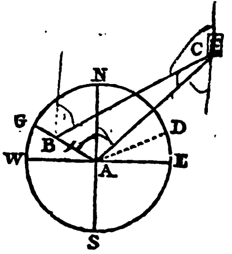
24. Supposing the three objects A, B, C, to be in a right line, as below; and that their distances are A C 1297, A B 7022, and B C 5725 yards, the angle A D C being 19° 10′ and B D C 84° 13′: required the distances D A, D C, and D B.

Ans. D A 3604.086, D C 2873.22, D B 5249.771.



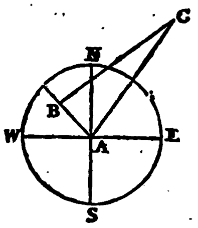
25. At a certain place in the county of Kent, the fort on Shooter's Hill bore from me N. E. 44¼°; and after going 15 miles in the direction N. W. 69½°, I perceived the fort again, which now bore N. E. 67¼°: required my distance from it at each station.

Ans. A C 26.5773 miles, and B C 35.4352 miles.

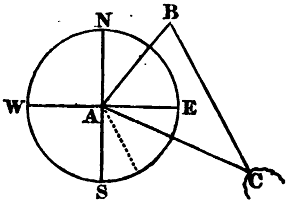


26. Being at a certain place A, St. Paul's church C, at London, bore from me N. E. 15¼°, and after travelling 20 miles further to B, in the direction N. W. 42½°, it bore N. E. 51½°: required my distance from it at the last place of observation.

Ans. B C 28.6052 miles.

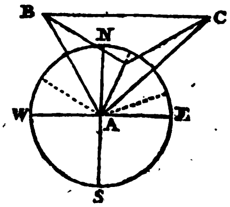


27. From a ship at sea A, I observed a point of land C to bear N. E. 102¼°, and after sailing 13 miles in the direction N. E. 40½° to B, it bore N. E. 141½°: required the distance of the last place of observation from the point of land. Ans. B C 18.099 miles.

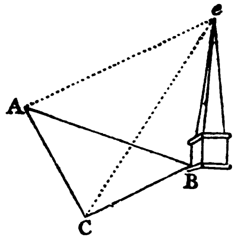


28. Coasting along shore, I observed two headlands, the 1st, B, bore N. W. 13½°, and the 2d, C, N. E. 36° 56′; then steering 12 miles in the direction N. E. 17° 52′, the 1st headland bore N. W. 68½°, and the 2d N. E. 70° 45′: required the bearing and distance of the two headlands from each other.

Ans. dist. B C 17.178 miles, and bearing of C from B, N. E. 95° 58½′.

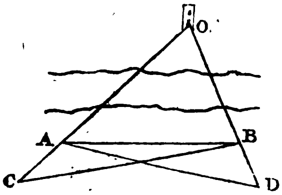


29. Let A, C, be any two points in the horizontal plane A B C, distant from each other 627 feet, and *e* B a distant spire, or other object; then, having the oblique ∠s *e* A C 57° 12′ *e* C A 49° 37′, taken at A and C, and the ∠s of elevation *e* A B 15° 12′, *e* C B 13° 45′: it is required to find the height of the object *e* B, its distance from each station, and the reduced angles A C B, B A C. Ans. ∠ A C B 48° 9′ 49″, ∠ B A C 55° 51′ 4″, A B 481.473, C B 534.811, height *e* B 130.867.



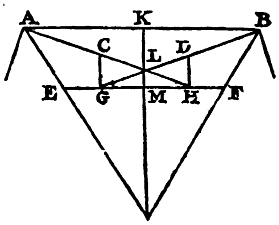
30. Wanting to know my distance from an object O, on the other side of a river, and having no instrument for taking angles, I took two stations A, B, 400 yards asunder, and then measured A C, B D, each 100 yards, in a direct line from the object: I also found the diagonal A D to be 450 yards, and B C 460: required the distance of O from each of the stations A, B.

Ans. A O 444.5657, B O 415.5711.



31. The side A B of a regular pentagon being 180 toises, the face of the bastion A C 50, and the perpen-

dicular K L 30, it is required to find, by trigonometrical calculation, all the other lines and angles of the fortification, supposing the lines of defence A H, B G, to be equal to the lines drawn from the salient angles A, B, to the shoulders D, C.



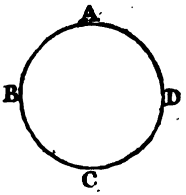
These lines and angles, when found, may be compared with those determined by construction in Muller's Elements of Fortification; or with a Table of the various dimensions of such plans, given by Robertson, at the end of his Navigation.

Note. A number of other questions might have been given, in this part of the work, relating to the reducing of angles, taken at a small distance from one of the stations, in certain geodetical operations, to be true angles at that station, which reduction becomes necessary when the centre of the instrument cannot be exactly placed in the vertical line occupied by the axis of a signal, and in other similar instances: but as the error arising from this cause, which seldom amounts to more than a few seconds, is commonly neglected, except in cases where the greatest accuracy is required, the reader is referred for further particulars on this head, to the works of *Cagnoli* and *Puissant*, before mentioned.

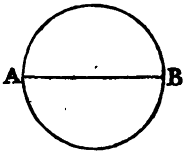
SPHERICAL TRIGONOMETRY.

Spherical Trigonometry is the science which treats of the properties and relations of spherical triangles, and of the methods of determining their sides and angles.

1. A sphere, or globe, is a solid contained under one uniform round surface, which is every where equally distant from a point within it, called its centre; as A B C D.

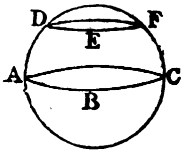


2. A diameter, or axis, of a sphere, is a right line passing through the centre, and terminated on each side by the convex surface; as A B.

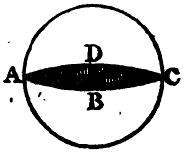


3. A great circle of the sphere, is that which divides the surface of it into two equal parts; and a small circle is that which divides it into two unequal parts.

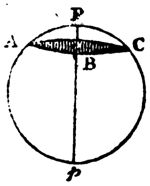
Thus A B C is a great circle of the sphere, and D E F a small circle (*t*).



4. Hence, also the plane of any great circle passes through the centre of the sphere, and divides the solid into two equal parts; as A B C D.



5. The poles of any circle, are the two extremities of that diameter, or axis, of the sphere, which is perpendicular to the plane of that circle; thus P, *p* are the poles of the circle A B C.



6. Hence, either pole of any circle is equidistant from every part of its circumference; and if it be a

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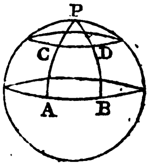
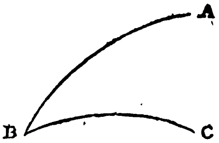
(*t*) Small circles of the sphere are not used in trigonometrical computations, on account of the diversity of their radii; but it may be observed, that any arc of a great circle, is to an arc of a small circle, of the same number of degrees, as the radius of the sphere is to the sine of the distance of the small circle from its pole. Thus A B : C D :: rad of the sphere : sine of P C. (fig. to def. 6.)

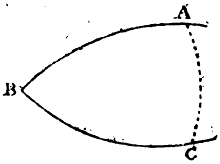
great circle, its pole is 90° from the circumference; thus if A B be an arc of a great circle, P A is equal to P B, being each quadrants; and in the small circle C D, P C is equal to P D.



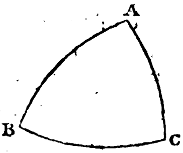
7. A spherical angle, is the inclination, or opening, of the arcs of two great circles of the sphere, which meet each other in a point on its surface; as A B C.



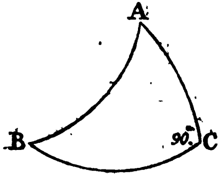
8. The measure of a spherical angle, is the arc of a great circle, drawn at the distance of 90° from its angular point, and intercepted by its two legs; thus if B A, B C, be quadrants, A C will be the measure of the angle A B C.



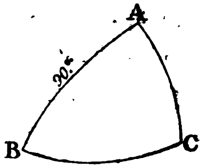
9. A spherical triangle, is a portion of the surface of a sphere, contained by the arcs of three great circles; as A B C.



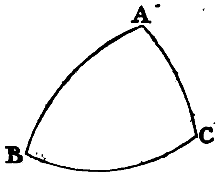
10. A right-angled spherical triangle, is that which has a right angle, or one of 90°; as A B C.



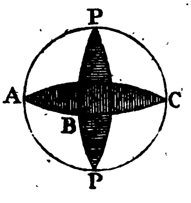
11. A quadrantal spherical triangle, is that which has one of its sides a quadrant, or 90°; as A B C.



12. An oblique-angled spherical triangle, is that which has each of its sides, or angles, greater or less than 90°; as A B C.



13. A great circle of the sphere is perpendicular to another circle, when its plane is perpendicular to the plane of that circle, and vice versâ; thus the circle P B P is perpendicular to A B C.



14. Any two sides, or angles, of a spherical triangle are said to be *like*, or *of the same kind*, when they are each equal to, each greater, or each less than 90°.

15. And if one of the sides, or angles, be equal to, or greater than, 90°, and the other less, they are said to be *unlike*, or *of different kinds*.

AXIOMS.

1. Every section of a sphere, by a plane passing through it, is a circle.

2. The centre of a sphere is the centre of all its great circles, and its axis is the common section of the planes of all the great circles which pass through its two extremities.

3. A great circle can be drawn through any two points on the surface of a sphere.

4. All parallel circles of the sphere have the same poles; and no two great circles can have a common pole.

5. Any two great circles of the sphere cut each other twice, at the distance of 180°, and make the angles at each point of section equal.

6. A great circle, which passes through the poles of any other circle, cuts it at right angles; and, if a great circle cut any other circle at right angles, it will pass through its poles (u).

(u) Most of the principles, here laid down as axioms, will be rendered sufficiently evident, by considering the position and nature of the circles usually drawn on a common globe. The 6th in particular, which is, perhaps, not quite so obvious, may be readily conceived, from observing that all the meridians pass through the north and south poles, and are perpendicular to the equator, and to all the parallels of latitude.

GENERAL PROPERTIES OF SPHERICAL TRIANGLES.

1. Any side, or angle, of a spherical triangle, is less than a semicircle, or 180°.

2. The greater side of any spherical triangle, is opposite to the greater angle, and the less side to the less angle.

3. The sum of any two sides of a spherical triangle, is greater than the third side; and their difference is less than the third side.

4. The difference of any two sides of a spherical triangle is less than a semicircle, or 180°; and the sum of the three sides is less than 360°.

5. The sum of the three angles of any spherical triangle, is greater than two right-angles, or 180°; and less than six right-angles, or 540°.

6. The sum of any two angles of a spherical triangle, is greater than the supplement of the third angle.

7. If the three sides of a spherical triangle be equal to each other, the three angles will also be equal; and vice versâ.

8. If any two sides of a spherical triangle be equal to each other, their opposite angles will also be equal, and vice versâ.

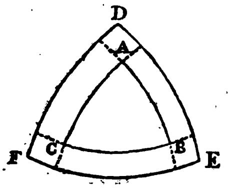
9. If the sum of any two sides of a spherical triangle be equal to 180°, the sum of their opposite angles will also be equal to 180°; and vice versâ.

10. If the three angles of a spherical triangle be all acute, or all right, or all obtuse, the three sides will be, accordingly, all less than 90°, or all equal to 90°, or all greater than 90°; and vice versâ.

11. Half the sum of any two sides of a spherical triangle, is of the same kind as half the sum of their opposite angles; or the sum of any two sides is of the same kind, with respect to 180°, as the sum of their opposite angles.

To these may also be added the following properties of the polar triangle; by which the data, in any case, may be changed from sides to angles, and from angles to sides.

If three arcs of great circles be described from the angular points A, B, C, of any spherical triangle ABC, as poles, or at 90° distance from them, the sides and angles of the new triangle DFE, so formed, will be the supplements of the opposite angles and sides of the other; and vice versâ.



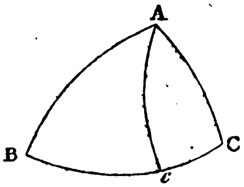
Thus, $DE=180^\circ-C$; $EF=180^\circ-A$; $FD=180^\circ-B$; and $D=180^\circ-BC$; $E=180^\circ-AC$; $F=180^\circ-AB$. Also, $AB=180^\circ-F$; $BC=180^\circ-D$; $AC=180^\circ-E$: and $A=180^\circ-FE$; $B=180^\circ-FD$; $C=180^\circ-DE$ (v).

OF THE AMBIGUOUS CASES OF SPHERICAL TRIANGLES.

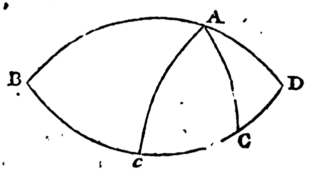
Any three of the six parts of a spherical triangle being given, the rest may always be found; except that in what are usually called *the two ambiguous cases*, the data are sometimes insufficient for limiting the triangle.

(v) The discovery of this useful property of the polar triangle is commonly attributed to Lansberg, who first gave it in his *Geometria Triangulorum*, published in 1591.

These cases are, when two sides and an angle opposite to one of them, or two angles and a side opposite to one of them, are given, to find the rest; in which instances there may be two triangles having the same data; and, consequently, if there be no other restriction or limitation, either of them will answer the conditions of the question.



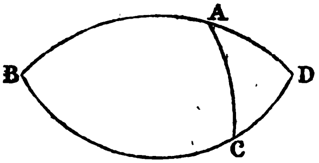
Thus, in the triangle A B C, let there be given the two sides A B, A C, and an opposite angle B; then if A *c* can be drawn equal to A C, there will, evidently, be two triangles, A B C, and A B *c*, which have the same given parts; whence the other angles may be either A C B, (=A *c* C) or its supplement A *c* B; or B A C, or B A *c*; and the remaining side will be B C or B *c*.



Also, in the triangle A B C, let there be given the two angles B and A C B, and an opposite side A C; then if A *c* can be drawn equal to A C, and B A, B C, be continued till they meet in D, there will be two triangles A B C, and A D *c*, which have the same given parts; whence the other sides may be either A B, or its supplement A D; or B C or D *c*; and the remaining angle will be B A C or D A *c*.

But as A *c* cannot, in all cases, be made equal to A C, the question may be limited, or not, according to the conditions of the data; which circumstance may be readily known, by considering that half the sum of any two sides of a spherical triangle is of the same kind as half the sum of their opposite angles, and taking the triangle accordingly.

Or, since the greater side of every spherical triangle is opposite to the greater angle, if only one of the values of the angle or side first found, or its supplement, agrees with this theorem, the triangle is limited, being that to which this value belongs. But if both the values are in conformity with the rule, the triangle is ambiguous (*w*).



(*w*) *Delambre*, p. 469 *Trig. de Cagnoli*, 1st edit. and p. 100 *Tables des Log. de Borda*, has given a theorem for determining, *à priori*, whether the Δ be ambiguous or not; but it is easy to show that the rule is not general.

Legendre also, p. 402 *Elémens de Géom.* 4th edit. and *Lacroix*, p. 68, *Elémens de Trig.* 2d edit. have pointed out all the cases which furnish either one or two solutions; but they are too

If the triangle A B C be right-angled, or quadrantal, there will be only one ambiguous case; which is when a side A C and its opposite angle B or D are given, to find the rest. For, if C be a right angle, it is plain that the hypothenuse may be either A B, or its supplement A D, and the remaining side and angle either B C and B A C, or their supplements C D and C A D.

Also, if A B be a quadrant, it is evident that the hypothenusal angle may be either A C B, or its supplement A C D, and the remaining side and angle either B C and B A C, or their supplements C D and C A D. And as the greater side in each of the triangles A B C, A D C, will be opposite to the greater angle, the question, in either of these cases, will always admit of two solutions.

OF RIGHT-ANGLED SPHERICAL TRIANGLES.

The different cases or varieties that may happen in the solution of right-angled spherical triangles, in which two things, together with the right angle, are always given, to find a fourth, are, in all, sixteen. But if these be restricted to such as depend upon the same principles, they may be reduced to six; or, when properly combined, to the three following general formulæ, which can be more easily remembered than if they were expressed separately:

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numerous to be remembered without reference to the table; and are, indeed, of little practical use, as the ambiguity, when an example is proposed in numbers, can always be readily discovered from the known affections of the sides and angles.

$$1.\ \text{r} \times \text{sin eith. leg} = \begin{cases} \text{sin its opp. } \angle \times \text{ sin hyp.} \\ \qquad\qquad \text{or} \\ \text{cot its adj}^{\text{t}}.\ \angle \times \text{ tan other leg.} \end{cases}$$

$$2.\ \text{r} \times \text{cos eith. } \angle = \begin{cases} \text{cos its opp. leg} \times \text{ sin other } \angle \\ \qquad\qquad \text{or} \\ \text{tan its adj}^{\text{t}}.\text{ leg} \times \text{ cot hyp.} \end{cases}$$

$$3.\ \text{r} \times \text{cos hyp} = \begin{cases} \text{cos one leg} \times \text{ cos other leg} \\ \qquad\qquad \text{or} \\ \text{cot one } \angle \times \text{ cot other } \angle. \end{cases}$$

And, in order to apply these forms to every case of right-angled spherical triangles, by converting them into analogies, it will be sufficient to remark, that any one of the terms on one side of the equation, is to either of the terms on the other side, as the remaining one of the latter is to the remaining one of the former; observing, at the same time, that when a leg or an angle is said to be opposite to another angle or leg, it will be adjacent to the remaining one; and vice versâ (*x*).

AFFECTIONS OF RIGHT-ANGLED SPHERICAL TRIANGLES.

1. The legs are of the same kind as their opposite angles; and conversely.

(*x*) In plane trigonometry, the knowledge of the three angles is only sufficient for determining the ratio of the three sides, and not their absolute values. But, in spherical trigonometry, where the sides are all arcs of great circles, they can be obtained from the three angles, without any other data.

Another remarkable difference between plane and spherical trigonometry is, that in the former, the third angle may always be determined from the other two; whereas in the latter, all the three angles are independent of each other, and must, therefore, be found separately.

2. The hypothenuse is less or greater than 90°, according as a leg and its adjacent angle, or the two legs, or the two angles, are like or unlike.

3. A leg is less or greater than 90°, according as its adjacent angle and the hypothenuse, or the other leg and the hypothenuse, are like or unlike.

4. An angle is acute or obtuse, according as its adjacent leg and the hypothenuse, or the other angle and the hypothenuse, are like or unlike.

OTHER PROPERTIES OF RIGHT-ANGLED SPHERICAL TRIANGLES.

1. If the hypothenuse be 90°, one of the legs and its opposite angle will be each 90°; and the other leg and angle will be both measured by the same number of degrees.

2. And if a leg, or an angle, be 90°, the opposite angle, or leg, and the hypothenuse, will be each 90°; and the other leg and angle will be both measured by the same number of degrees.

3. If a leg be less than the hypothenuse, their sum will be less than 180°; and if it be greater than the hypothenuse, their sum will be greater than 180° (*y*).

(*y*) Properties similar to this, and the following one, are given by *Cagnoli*, p. 244, *Traité de Trig.*, and by *Maskelyne*, in his Introduction to *Taylor's Logarithms;* but they are so expressed, in each of these works, that, if followed without any other restriction, they would frequently lead to an impossible triangle. The same observation may also be applied to the two corresponding properties of quadrantal spherical triangles.

4. If a leg be less than its opposite angle, their sum will be less than 180°; and if it be greater than its opposite angle, their sum will be greater than 180°.

5. The difference of the two oblique angles is less than 90°; and their sum is greater than 90°, and less than 270°.

6. The three sides are either all equal to, or all less than, 90°; or two of them are greater than 90°, and the other less (z).

The six cases of right-angled spherical triangles, before mentioned, may be ranged as follows:

Given
A leg and its opp. ∠
A leg and its adj^{t}. ∠
The hyp. and a leg
The hyp. and an ∠
The two legs
The two ∠^{s}
to find the other parts.

CASE I.

When a leg and its opposite angle are given, to find the rest.

1. *To find the other leg.*

As rad : tan giv. leg : : cot opp. or giv. ∠ : sin other leg.

Which leg may be either an arc less than 90°, or its supplement.

(z) A right-angled spherical triangle may have either,
1. One right ∠, and two acute, or two obtuse ∠^{s};
2. Or two right ∠^{s}, and one acute, or one obtuse ∠;
3. Or all its three ∠^{s} may be right ∠^{s}.

2. *To find the other ∠.*

As cos given leg : cos opp. or giv. ∠ : : rad : sin other ∠.

Which ∠ may be either an acute ∠, or its supplement.

3. *To find the hypothenuse.*

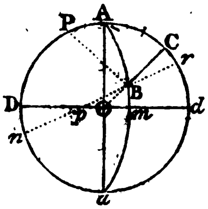
As sin giv. ∠ : sin opp. or giv. leg : : rad : sin hyp.

Which hyp. may be either an arc less than 90°, or its supplement.

EXAMPLES.

1. In the right-angled spherical triangle A B C, having the leg B C 42° 25′, and its opposite ∠ A 46° 30′, to find the rest.

BY CONSTRUCTION.



1. Describe the circle A D *a d* with the chord of 60°; and draw the diameters A *a*, D *d*, at right angles to each other.

2. Set the semitangent of the complement of the angle A (43° 30′) from o to *m*, and through the three points A, *m*, *a*, describe a circle.

3. From o as a centre, with the semitangent of the complement of B C (47° 35′) as a radius, describe an arc, cutting the circle A *m a* in B.

4. Through the points o, B, draw the line o B C; and A B C, or a B C, will be the Δ required, each having the same data; which shows this case to be ambiguous.

To measure the required parts.

1. Set off the semitangent of the given ∠ A (46° 30′) from o to *p*; and take C P equal to the chord of 90°.

2. Through the points B, *p*, draw the line *n p* B *r*, cutting the circle in *n* and *r*.

3. Then P *n*, taken on the scale of chords, gives the ∠ B 68° 48′, A *r*, taken on the same scale, gives A B 68° 25′, and A C on the same scale is 60° 6′.

BY CALCULATION.

:	Rad, or sin	90°	10.0000000
:	Tan B C	42° 25′	9.9607842
: :	Cot ∠ A	46° 30′	9.9772500
:	Sin A C	60° 6′ 55″ or 119° 53′ 5″	9.9380342
:	Cos B C	42° 25′	9.8682088
:	Cos ∠ A	46° 30′	9.8378122
: :	Rad, or sin	90°	10.0000000
:	Sin ∠ B	68° 48′ 45″ or 111° 11′ 15″	9.9696034
:	Sin ∠ A	46° 30′	9.8605622
:	Sin B C	42° 25′	9.8289930
: :	Rad, or sin	90°	10.0000000
:	Sin A B	68° 25′ 3″ or 111° 34′ 57″	9.9684308

INSTRUMENTALLY.

1. Extend the compasses from 46° 30′ to 42° 25′ on the line of tangents, and that extent will reach, on the sines, from 90° to 60° 6′, the side A C.

2. Extend from 47° 35′ (the compᵗ. of B C) to 43° 30′ (the compᵗ. of ∠ A) on the sines, and that extent will reach, on the same line, from 90° to 68° 48′ the ∠ B.

3. Extend from 46° 30′ (∠ A) to 42° 25′ (B C) on the sines, and that extent will reach, on the same line, from 90° to 68° 25′, the side A B (*a*).

2. In the right-angled spherical triangle A B C,

Gⁿ. { The leg $\overset{a}{\text{BC}}$ 15° 9′
Its opp. ∠ A 24° 17′
Required the other parts.

Ans. { $\overset{b}{\text{AC}}$ 36° 52′ 44″ or 143° 7′ 16″
∠ B 70° 47′ 40″ or 109° 12′ 20″
$\overset{c}{\text{AB}}$ 39° 27′ 24″ or 140° 32′ 36″

3. In the right-angled spherical triangle A B C,

Gⁿ. { The leg $\overset{a}{\text{BC}}$ 97° 20′
Its opp. ∠ A 84° 45′
Required the other parts.

Ans. { $\overset{b}{\text{AC}}$ 45° 33′ 39″ or 134° 26′ 21″
∠ B 45° 47′ 46″ or 184° 12′ 14″
$\overset{c}{\text{AB}}$ 84° 52′ 22″ or 95° 7′ 38″

(*a*) In following the three general rules, which have been given for right-angled spherical triangles, the proportion used in the calculation is not always that which is adapted to the instrumental solution; but the former may be easily reduced to the latter by proper substitutions: Thus, since rad : tan B C : : cot A : sin A C by the first formula, if $\frac{r^2}{\tan A}$ be put in the place of its equal, cot A, the proportion will become tan. A : tan B C : : rad : sin A C, which is that applied to the instrument; and, if thought necessary, will equally serve for the numeral solution.

It may also be observed, that, in the instrumental computation, when the extent on the tangents reaches beyond the line, it must be set as far back as it reaches over; the method of doing which may be seen in the solution of case IV. following; where it is more fully described.

4. In the right-angled spherical triangle A B C,

Gⁿ. { The leg $\overset{b}{\text{A C}}$ 38° 51′ / Its opp. ∠ B 42° 51′ Ans. { $\overset{a}{\text{B C}}$ 60° 15′ 40″ or 119° 44′ 20″ / ∠ A 70° 17′ 3″ or 109° 42′ 57″ / $\overset{c}{\text{A B}}$ 67° 16′ 28″ or 112° 43′ 32″
Required the other parts.

Note. Several of the problems, given in this part of the work, may often be more conveniently resolved by means of some of the formulæ in the table of cases, page 156 et seq., where the limits of the data, and other circumstances, are more particularly pointed out.

CASE II.

When a leg and its adjacent angle are given, to find the rest.

1. *To find the other leg.*

As cot given ∠ : sin adjacent, or given leg : : rad : tan other leg.

Which leg is like its opposite ∠.

2. *To find the other* ∠.

As rad : sin given ∠ : : cos adjacent, or given leg : cos other ∠.

Which ∠ is like its opposite leg.

3. *To find the hypothenuse.*

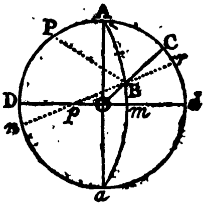
As tan given leg : cos adjacent, or given ∠ : : rad : cot hyp.

Which hyp. is less than 90° if the given leg and ∠ are like; but greater than 90° if they are unlike.

EXAMPLES.

1. In the right-angled spherical triangle, A B C, having the leg A C 54° 46′, and its adjacent angle A 47° 56′, to find the rest.

BY CONSTRUCTION.



1. Describe the circle A D *a d* with the chord of 60°; and draw the diameters A *a*, D *d*, at right angles to each other.

2. Set the semitangent of the complement of the angle A (42° 4′) from O to *m*; and through the points A, *m*, *a*, describe a circle.

3. Set off A C (54° 46′) by the scale of chords, and draw C O, cutting the circle A *m a* in B; then A B C will be the triangle required.

To measure the required parts.

1. Set off the semitangent of the given ∠ A (47° 56′) from O to *p*, and make C P equal to the chord of 90°.

2. Through the points B, *p*, draw the line *n p* B *r*, cutting the circle in *r* and *n*.

3. Then P *n*, taken on the scale of chords, gives the ∠ B 64° 38′, A *r*, on the same scale, gives A B 64° 40′, and O B, taken on the line of semitangents, and then subtracted from 90°, gives B C 42° 8′.

BY CALCULATION.

: Cot ∠ A - - - - 47° 56′ - - 9.9554535

: Sin A C - - - - - 54° 46′ - - 9.9121207

:: Rad, or sin - - - 90° - - - - 10.0000000

: Tan B C - - - - 42° 8′ 46″ 9.9566672

Which side is acute, being like its opposite ∠ A.

: Rad, or sin - - 90° - - - - - 10.0000000
: Sin ∠ A - - - 47° 56′ - - - 9.8706179
: : Cos A C - - - 54° 46′ - - - 9.7611063
: Cos ∠ B - - - 64° 38′ 31″ - 9.6317242

Which ∠ is acute, being like its opposite leg A C.

: Tan A C - - - 54° 46′ - - - 10.1510145
: Cos ∠ A - - - 47° 56′ - - - 9.8260715
: : Rad, or sin - - 90° - - - - - 10.0000000
: Cot A B - - - - 64° 40′ 34″ - 9.6750570

Which side is less than 90°, because A C and ∠ A are like.

INSTRUMENTALLY.

1. Extend the compasses from 90° to 54° 46′ (A C) on the lines of sines; and this extent will reach, on the tangents, from 47° 56′ (∠ A) to 42° 8′, the leg B C.

2. Extend from 90° to 47° 56′ (∠ A) on the sines, and this extent will reach, on the same line, from 35° 14′ (compt. of A C) to 25° 22′, the complement of ∠ B.

3. Extend from 90° to 42° 4′ (compt. of ∠ A) on the sines, and this extent will reach, on the tangents, from 35° 14′ (compt. of A C) to 25° 20′, the complement of A B (*b*).

(*b*) If $\frac{r^2}{\tan A}$ be substituted for cot A in the first stating of the numeral calculation of this case, it will become rad : sin A C : : tan A : tan B C; and if $\frac{r^2}{\cot AC}$ be put for tan A C, in the third stating, it will become rad : cos A : : cot A C : cot A B; which are the proportions adapted to the instrumental solution.

2. In the right-angled spherical triangle A B C,

Given { The leg A C 36° 52′
Its adj^{t}. ∠ A 24° 17′
Required the other parts.

Ans. { B C 15° 8′ 45″
∠ B 70° 47′ 28″
A B 39° 26′ 40″

3. In the right-angled spherical triangle A B C,

Given { The leg A C 76° 20′
Its adj^{t}. ∠ A 91° 15′
Required the other parts.

Ans. { B C 91° 17′ 11″
∠ B 76° 20′ 12″
A B 90° 18′ 14″

4. In the right-angled spherical triangle A B C,

Given { The leg B C 119° 6′
Its adj^{t}. ∠ B 44° 37′
Required the other parts.

Ans. { A B 111° 36′ 46″
∠ A 109° 58′ 24″
A C 40° 46′

CASE III.

When the hypothenuse and a leg are given, to find the rest.

1. *To find the opposite* ∠.

As sin hyp. : rad : : sin giv. leg : sin its opp. ∠.
Which ∠ is like its opposite leg.

2. *To find the adjacent* ∠.

As rad : cot hyp. : : tan giv. leg. : cos its adj^{t}. ∠.

Which ∠ is acute, if the hyp. and given leg are like; but obtuse, if they are unlike.

3. *To find the other leg.*

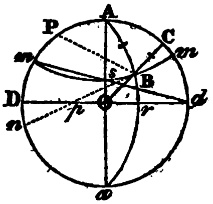
As cos giv. leg : rad : : cos hyp. : cos other leg.

Which leg is less that 90° if the hyp. and given leg are like; but greater than 90° if they are unlike.

EXAMPLES.

1. In the right-angled spherical triangle A B C, having the hypothenuse A B $65^\circ\ 50'$, and the leg B C $42^\circ\ 20'$, to find the rest.

BY CONSTRUCTION.



1. With the chord of 60° describe the circle A D *a d*, and draw the diameters A *a*, D *d*, at right angles to each other.

2. Set off the hyp. A B ($65^\circ\ 50'$), by a scale of chords, each way, from A to *m*; and draw *d m*, cutting A *a* in *s*.

3. Through the points *m*, *s*, *m*, describe a circle; and from O, with the semitangent of the complement of B C ($47^\circ\ 40'$) as a radius, intersect the former circle in B.

4. Describe a circle through the points A, B, *a*, and draw O B C; then A B C will be the triangle required.

To measure the required parts.

1. Measure the distance O *r*, in degrees, on a scale of semitangents, and set off its complement on the same scale, from O to *p*.

2. Through the points B, *p*, draw the line B *p n*, cutting the circle in *n*; and take C P equal to the chord of 90°.

3. Then P *n*, on the scale of chords, gives ∠ B 65° 52′, O *r*, taken on the scale of semitangents, and then subtracted from 90°, gives ∠ A 47° 34′, and A C, on the scale of chords, is 56° 22′.

BY CALCULATION.

:	Sin A B - - - -	65° 50′ - - -	9.9601655
:	Rad, or sin - -	90° - - - - -	10.0000000
: :	Sin B C - - - -	42° 20′ - - -	9.8283006
:	Sin ∠ A - - -	47° 34′ 20″	9.8681351

Which ∠ is acute, being like its opposite leg B C.

:	Rad, or sin - -	90° - - - - -	10.0000000
:	Cot A B - - - -	65° 50′ - - -	9.6519742
: :	Tan B C - - - -	42° 20′ - - -	9.9595156
:	Cos ∠ B - - - -	65° 52′ 20″ -	9.6114897

Which ∠ is acute, because A B and ∠ B are like.

:	Cos B C - - - -	42° 20′ - - -	9.8687851
:	Rad, or sin - -	90° - - - - -	10.0000000
: :	Cos A B - - - -	65° 50′ - - -	9.6121397
:	Cos A C - - - -	56° 25′ 20″	9.7433546

Which side is less than 90°, because A B and B C are like.

INSTRUMENTALLY.

1. Extend the compasses from 65° 50′ (A B) to 90° on the sines, and that extent will reach, on the same line, from 42° 20′ (B C) to 47½°, the ∠ A.

2. Extend from 65° 50′ (A B) to 42° 20′ (B C), on the tangents, and that extent will reach, on the sines, from 90° to 24° 8′, the complement of ∠ B.

3. Extend from 47° 40′ (compᵗ. of B C) to 90° on the sines, and that extent will reach, on the same

line, from 34° 10′ (compt. of A B) to 24° 8′, the complement of A C (c).

2. In the right-angled spherical triangle A B C,

Given	The hyp. A B 37° 25′	Ans.	∠ A 21° 59′ 18″
	The leg B C 13° 9′		∠ B 72° 13′ 7″
	To find the other parts.		A C 35° 21′ 4″

3. In the right-angled spherical triangle A B C,

Given	The hyp. A B 37° 10′	Ans.	∠ A 119° 59′ 29″
	The leg B C 148° 27′		∠ B 144° 5′ 8″
	Required the other parts.		A C 159° 14′ 42″

4. In the right-angled spherical triangle A B C,

Given	The hyp. A B 69° 27′	Ans.	∠ A 54° 6′ 48″
	The leg A C 57° 24′		∠ B 64° 7′ 9″
	Required the other parts.		B C 49° 20′ 35″

CASE IV.

When the hypothenuse and an angle are given, to find the rest.

1. *To find the opposite leg.*

As rad : sin hyp. : : sin giv. ∠ : sin its opp. leg.
Which leg is like its opposite ∠.

2. *To find the adjacent leg.*

As cot hyp. : rad : : cos giv. ∠ : tan its adjt. leg.
Which leg is less than 90° when hyp. and given ∠ are like; but greater than 90° when they are unlike.

(c) By substituting $\frac{r^2}{\tan \mathrm{A\,B}}$ for cot A B, in the second stating of the numerical solution, it will become tan A B : tan A C : : rad : cos B for the instrumental solution.

3. *To find the other* ∠.

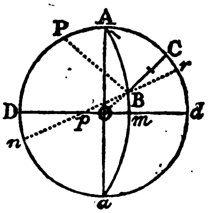
As cot giv. ∠ : rad : : cos hyp. : cot other ∠.

Which ∠ is acute when hyp. and given ∠ are like; but obtuse when they are unlike.

EXAMPLES.

1. In the right-angled spherical triangle A B C, having the hypothenuse A B 65° 5′, and the angle A 48° 12′, to find the rest.

BY CONSTRUCTION.



1. With the chord of 60° describe the circle A D a d, and draw the diameters A a, D d, at right angles to each other.

2. Set off o m equal to the semitangent of the complement of the ∠ A (41° 48′), and through the points A, m, a, describe a circle.

3. Set off the semitangent of the ∠ A (48° 12′) from o to p, and take A r, on the line of chords, equal to A B (65° 5′).

4. Through the points r, p, draw the line r B p n, cutting the circle A m a in B, and the former circle in n; then, through the points o, B, draw the line o C, and A B C will be the triangle required.

To measure the required parts.

1. Set off C P equal to the chord of 90°; then P *n*, taken on the scale of chords, will give ∠ B (64° 46′); and A C, on the same scale, is 55° 7′.

2. And if O B be taken on the line of semitangents, and then subtracted from 90°, it will give B C 42° 32′.

BY CALCULATION.

:	Rad, or sin	90°	10.0000000
:	Sin A B	65° 5′	9.9575697
: :	Sin ∠ A	48° 12′	9.8724337
:	Sin B C	42° 32′ 19″	9.8300034

Which side is acute, being like its opposite ∠ A.

:	Cot A B	65° 5′	9.6670214
:	Rad, or sin	90°	10.0000000
: :	Cos ∠ A	48° 12′	9.8238213
:	Tan A C	55° 7′ 32″	10.1567999

Which side is less than 90°, because A B and ∠ A are like.

:	Cot ∠ A	48° 12′	9.9513876
:	Rad, or sin	90°	10.0000000
: :	Cos A B	65° 5′	9.6245911
:	Cot ∠ B	64° 46′ 14″	9.6732035

Which ∠ is acute, because A B and ∠ A are like.

INSTRUMENTALLY.

1. Extend the compasses from 90° to 65° 5′ (A B) on the sines, and that extent will reach, on the same line, from 48° 12′ (∠ A) to 42° 32′, the leg B C.

2. Extend from 90°, on the sines, to 41° 48′ (compt. of ∠ A); then apply this extent from 45° on the tangents, towards the left hand, and, keeping the latter

point of the compasses fixed, turn the other leg, and extend it till it reaches to 65° 5′ (A B); which last extent will reach from 45° to 55° 7′, the leg A C.

3. Extend from 24° 55′ (compᵗ. of A B) to 90° on the sines; then apply this extent from 45° on the tangents, backwards, on the left hand, and keeping the latter point of the compasses fixed, turn the other leg till it reaches to 41° 48′ (compᵗ. ∠ A); and this extent will reach from 45° to 64° 46′, the ∠ B; the divisions, in this case, falling as much under 45° as they would have fallen beyond it (*d*).

2. In the right-angled spherical triangle A B C,

Given	The hyp. A B 39° 27′	Ans.	B C 16° 53′ 4″
	The ∠ A - - 27° 12′		A C 36° 11′ 58″
	Required the other parts.		∠ B 68° 21′ 17″

3. In the right-angled spherical triangle A B C,

Given	The hyp. A B 79° 25′	Ans.	B C 41° 22′ 14″
	The ∠ A - - 42° 15′		A C 75° 50′
	Required the other parts.		∠ B 80° 31′ 43″

4. In the right-angled spherical triangle A B C,

Given	The hyp. A B 98° 20′	Ans.	B C 105° 20′ 11″
	The ∠ B - - 57° 43′		A C 56° 46′ 17″
	Required the other parts.		∠ A 102° 55′ 14″

(*d*) By putting $\frac{r^2}{\tan AB}$ for cot A B in the second numeral stating, it will become rad : cos A : : tan A B : tan A C; and if $\frac{r^2}{\tan B}$ be put for cot B in the third stating, it will become cos A B : rad : : cot A : tan B; which are the analogies adapted to the instrumental solution.

CASE V.

When the two legs are given, to find the rest.

1. *To find either of the* ∠s.

As tan one of the legs : rad : : sin other leg : cot its adjacent ∠.

Which ∠ is like its opposite leg.

2. *To find the hypothenuse.*

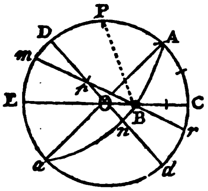
As rad : cos either leg : : cos other leg : cos hyp.

Which hyp. is less than 90° if the legs are like; but greater than 90° if they are unlike.

EXAMPLES.

1. In the right-angled spherical triangle A B C, having the leg A C 52° 13′, and the leg B C 42° 17′, to find the rest.

BY CONSTRUCTION.



1. From O, as a centre, with the chord of 60° describe a circle, and draw the diameter E C.

2. Set off C A (52° 13′) from the scale of chords, and make O B equal to the semitangent of the complement of B C (47° 43′).

3. Draw the diameters A *a*, D *d*, at right angles to each other, and through the points A, B, *a*, describe a circle; then B B C will be the triangle required.

To measure the required parts.

1. Take the measure of o *n* in degrees, on the scale of semitangents, and set its complement, on the same scale, from o to *p*.

2. Through *p*, B, draw the line *mp*B*r*, cutting the circle in *m* and *r*; and set off C P equal to the chord of 90°.

3. Then P *m*, taken on the line of chords, gives the ∠ B 62° 27′; A *r*, on the same line, gives A B 63° 2′; and o *n*, taken on the line of semitangents, and then subtracted from 90°, gives ∠ A 49°.

BY CALCULATION.

:	Tan A C - - - - -	52° 13′ - -	10.1105786
:	Rad, or sin - - -	90° - - - -	10.0000000
: :	Sin B C - - - - - -	42° 17′ - -	9.8278843
:	Cot ∠ B - - - - -	62° 27′ 19″	9.7173057

Which ∠ is acute, being like its opposite leg A C.

:	Tan B C - - - - -	42° 17′ - -	9.9587542
:	Rad, or sin - - -	90° - - - -	10.0000000
: :	Sin A C - - - - - -	52° 13′ - -	9.8978103
:	Cot ∠ A - - - - -	49° 0′ 25″	9.9390561

Which ∠ is acute, being like its opposite leg B C.

:	Rad, or sin - - -	90° - - - -	10.0000000
:	Cos A C - - - - -	52° 13′ - -	9.7872317
: :	Cos B C - - - - -	42° 17′ - -	9.8691301
:	Cos A B - - - - -	63° 2′ 45″	9.6563618

Which side is less than 90°, because A C and B C are like.

INSTRUMENTALLY.

1. Extend the compasses from 42° 17′ (B C) to 90° on the sines, and this extent will reach, on the tangents, from 52° 13′ (A C) to 62° 27′, the ∠ B (e).

2. Extend from 52° 13′ (A C) to 90°, on the sines, and this extent will reach, on the tangents, from 42° 17′ (B C) to 49°, the ∠ A.

3. Extend from 90°, on the sines, to 37° 47′ (compᵗ. of A C), and this extent will reach, on the same line, from 47° 43′ (compᵗ. of B C) to 26° 58′, the complement of A B.

2. In the right-angled spherical triangle A B C,

Given { The leg $\overset{\frown}{AC}$ 29° 17′ / The leg $\overset{\frown}{BC}$ 49° 27′ Ans. { ∠ A 67° 17′ 28″ / ∠ B 36° 25′ 44″ / $\overset{\frown}{AB}$ 55° 27′ 23″

Required the other parts.

3. In the right-angled spherical triangle A B C,

Given { The leg $\overset{\frown}{AC}$ 76° 25′ / The leg $\overset{\frown}{BC}$ 37° 31′ Ans. { ∠ A 38° 18′ 17″ / ∠ B 81° 37′ 46″ / $\overset{\frown}{AB}$ 79° 15′ 51″

Required the other parts.

4. In the right-angled spherical triangle A B C,

Given { The leg $\overset{\frown}{AC}$ 97° 29′ / The leg $\overset{\frown}{BC}$ 104° 12′ Ans. { ∠ A 104° 5′ 2″ / ∠ B 97° 15′ 25″ / $\overset{\frown}{AB}$ 38° 10′ 10″

Required the other parts.

(e) By substituting $\frac{r^2}{\tan A}$ for cot ∠ B in the first logarithmic stating, and $\frac{r^2}{\tan A}$ for cot A in the second, the two analogies become sin B C : rad : : tan A C : tan B, and sin A C : rad : : tan B C : tan A; which are those used in the instrumental solution.

✓ CASE VI.

When the two oblique angles are given, to find the rest.

1. *To find either of the legs.*

As sin one of the given ∠ : rad : : cos other ∠ : cos its opposite leg.

Which leg is like its opposite ∠.

2. *To find the hypothenuse.*

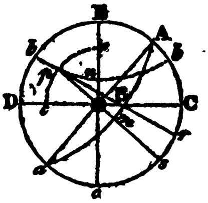
As rad : cot either ∠ : : cot other ∠ : cos hyp.

Which hyp. is less than 90° when the given ∠s are like; but greater than 90° when they are unlike.

EXAMPLES.

1. In the right-angled spherical triangle ABC, having the angle A 48° 13′, and the angle B 64° 27′, to find the rest.

BY CONSTRUCTION.



1. Describe the circle E D *e* C with the chord of 60°; and draw the diameters E *e*, D C, at right angles to each other.

2. From E, with the chord of ∠ B (64° 27′), set off, each way, E *b*, E *b*; and from O take O *n* equal to the semitangent of the complement of that angle.

3. Through the points *b*, *n*, *b*, describe a circle; and from O, with the semitangent of ∠ A (48° 13′) describe the arc *e e*, cutting the former in *p*.

4. Through *p* o, draw the diameter *p s*, and another A *a*, at right angles to it.

5. Set off o *m* equal to the semitangent of the complement of ∠ A (41° 47′); then through the points A, *m*, *a*, describe a circle, cutting D C in B; and A B C will be the triangle required.

To measure the required parts.

Through the points *p*, B, draw the line *p* B *r*, cutting the circle E D *e* C in *r*; then A *r*, taken on the chords, gives A B 64° 42′, A C on the same scale is 54° 39′, and O B, taken on the line of semitangents, and then subtracted from 90°, gives B C 42° 23′.

BY CALCULATION.

:	Sin ∠ A	48° 13′	9.8725466
:	Rad, or sin	90°	10.0000000
: :	Cos ∠ A	64° 27′	9.6347780
:	Cos A C	54° 39′ 42″	9.7622314

Which side is less than 90°, being like its opp. ∠ B.

:	Sin ∠ B	64° 27′	9.9553073
:	Rad, or sin	90°	10.0000000
: :	Cos ∠ A	48° 13′	9.8236800
:	Cos B C	42° 23′ 35″	9.8683727

Which side is less than 90°, being like its opp. ∠ A.

:	Rad, or sin	90°	10.0000000
:	Cot ∠ A	48° 13′	9.9511334
: :	Cot ∠ B	64° 27′	9.6794708
:	Cos A B	64° 42′ 42″	9.6306042

Which side is less than 90°, because ∠ ˢ A and B are like.

INSTRUMENTALLY.

1. Extend the compasses from 48° 13′ (∠ A) to 90° on the sines, and that extent will reach, on the same line, from 25° 33′ (compᵗ. of ∠ B) to 35° 21′, the complement of A C.

2. Extend from 64° 27′ (∠ B) to 90° on the sines, and that extent will reach, on the same line, from 41° 47′ (compᵗ. of ∠ A) to 47° 37′, the complement of B C.

3. Extend from 48° 13′ (∠ A) to 25° 33′ (compᵗ. of ∠ B) on the tangents, and this extent will reach, on the sines, from 90° to 25° 18′, the complement of A B (*f*).

2. In the right-angled spherical triangle A B C,

Given { The ∠ A 24° 30′ / The ∠ B 70° 25′ }
Required the other parts.
Ans. { A C 36° 4′ 28″ / B C 15° 1′ 17″ / A B 38° 40′ 52″ }

3. In the right-angled spherical triangle A B C,

Given { The ∠ A 39° 12′ / The ∠ B 82° 9′ }
Required the other parts.
Ans. { A C 77° 31′ 12″ / B C 38° 31′ 51″ / A B 80° 16′ 2″ }

4. In the right-angled spherical triangle A B C,

Given { The ∠ A 105° 8′ / The ∠ B 31° 20′ }
Required the other parts.
Ans. { A C 27° 46′ 5″ / B C 120° 8′ 5″ / A B 116° 22′ 25″ }

(*f*) The analogy for obtaining the value of A B by the instrument, is got by substituting $\frac{r^2}{\tan A}$ for cot A, in the numeral solution, which then becomes tan A : cot B : : rad : cos A B.

OF QUADRANTAL SPHERICAL TRIANGLES.

The different cases, or varieties, that may happen in the solution of quadrantal spherical triangles, in which two things, together with the quadrantal side, are always given, to find a third, are the same as in right-angled spherical triangles.

And since the sides and angles of any quadrantal spherical triangle are the supplements of the opposite angles and sides of a right-angled spherical triangle, described from its angular points as poles, the three general formulæ which have been given for the latter, may be readily converted into the following ones, which are equally applicable to all the cases of quadrantal spherical triangles:

$$1.\ r \times \text{sin eith. } \angle = \left\{ \begin{array}{c} \text{sin its opp. side} \times \text{sin hyp}^{l}.\ \angle, \\ \text{or} \\ \text{cot its adj}^{t}.\ \text{side} \times \text{tan other } \angle. \end{array} \right.$$

$$2.\ r \times \text{cos eith. side} = \left\{ \begin{array}{c} \text{cos its opp. } \angle \times \text{sin other side}, \\ \text{or} \\ -\text{tan its adj}^{t}.\ \angle \times \text{cot hyp}^{l}.\ \angle. \end{array} \right.$$

$$3.\ r \times \text{cos hyp}^{l}.\ \angle = \left\{ \begin{array}{c} -\text{cos one } \angle \times \text{cos other } \angle, \\ \text{or} \\ -\text{cot one side} \times \text{cot other side}. \end{array} \right.$$

Where it is to be remarked, that the angle opposite the quadrantal side is called the hypothenusal angle, and the other parts simply the sides and angles. And in applying these forms to practice, it is only necessary to attend to the observations that have been already made for right-angled spherical triangles.

AFFECTIONS OF QUADRANTAL SPHERICAL TRIANGLES.

1. The sides are of the same kind as their opposite angles; and conversely.

2. The hypothenusal angle is greater or less than 90°, according as a side and its adjacent angle, or the two sides, or the other two angles, are like or unlike.

3. An angle at the quadrant is obtuse or acute, according as its adjacent side and the hypothenusal angle, or the other angle and hypothenusal angle, are like or unlike.

4. A side is greater or less than 90°, according as its adjacent angle and the hypothenusal angle, or the other side and the hypothenusal angle, are like or unlike.

OTHER PROPERTIES OF QUADRANTAL SPHERICAL TRIANGLES.

1. If the hypothenusal angle be 90°, one of the other angles and its opposite side will be each 90°, and the other side and angle will be both measured by the same number of degrees.

2. If an angle, or a side, be 90°, the opposite side, or angle, and the hypothenusal angle will be each 90°; and the other angle and side will be both measured by the same number of degrees.

3. If an angle at the quadrant be less than the hypothenusal angle, their sum will be less than 180°; and if it be greater than the hypothenusal angle, their sum will be greater than 180°.

4. If a side be greater than its opposite angle, their sum will be less than 180°; and if it be less than its opposite angle, their sum will be greater than 180°.

5. The difference of the two sides is less than 90°; and their sum is greater than 90°, and less than 270°.

6. The three angles are either all equal to, or all less than 90°; or two of them are greater than 90°, and the other less.

The six cases of quadrantal spherical triangles already mentioned, may be ranged as follows:

Given
- A side and its opp. ∠
- A side and its adj^{t}. ∠
- The hyp^{l}. ∠ and a side
- The hyp^{l}. ∠ and another ∠
- The two sides
- The two ∠'

to find the other parts.

CASE I.

When a side and its opposite angle are given, to find the rest.

1. *To find the other ∠.*

As rad : cot giv. side : : tan giv. ∠ : sin other ∠.

Which ∠ may be either an acute ∠ or its sup^{t}.

2. *To find the other side.*

As cos giv ∠ : cos opp. or giv. side : : rad : sin other side.

Which side may be an arc less than 90°, or its sup^{t}.

3. *To find the hypothenusal ∠.*

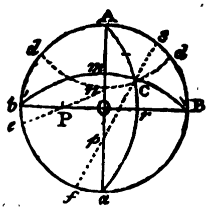
As sin giv. side : sin opp. or giv. ∠ : : rad : sin hyp^{l}. ∠.

Which hypothenusal ∠ may be either an acute ∠, or its supplement.

EXAMPLES.

1. In the quadrantal spherical triangle A B C, having A C 63° 58′, and its opposite angle B 54° 19′, to find the rest.

BY CONSTRUCTION.



1. Describe the circle A *b* *a* B with the chord of 60°, and draw the diameters A *a*, B *b*, at right angles to each other.

2. Set off A *d*, A *d*, each way, with the chord of A C (63° 58′), and take O *n* equal to the semitangent of the complement of that arc (26° 2′), and O *m* equal to the semitangent of the complement of ∠ B (35° 41′).

3. Through the points B, *m*, *b*, and *d*, *n*, *d*, describe two circles, cutting each other in C; then if a circle be described through the points A, C, *a*, the triangle A B C, or *b* A C, will be the one required, each having the same data; which shows the case to be ambiguous.

To measure the required parts.

1. Take O *r* on the scale of semitangents, and set off its complement, on the same scale, from O to P.

2. Make O *p* equal to the semitangent of the ∠ B (54° 19′), and through the points C P, C *p*, draw the lines C *e*, and C *f*.

3. Then B *s*, taken on the scale of chords, gives B C 48° 48′; *e f*, on the same scale, gives ∠ C 64° 41′; and the complement of the degrees in O *r*, taken on the semitangents, gives ∠ A 42° 51′.

BY CALCULATION.

:	Cos ∠ B	54° 19′	9.7658957
:	Cos A C	63° 58′	9.6423596
: :	Rad, or sin	90°	10.0000000
:	Sin B C 48° 48′ 3″ or 131° 11′ 57″		9.8764639

:	Rad, or sin	90°	10.0000000
:	Tan ∠ B	54° 19′	10.1437958
: :	Cot A C	63° 58′	9.6888227
:	Sin ∠ A 42° 51′ 25″ or 137° 8′ 35″		9.8326185 (g)

:	Sin A C	63° 58′	9.9535369
:	Sin ∠ B	54° 19′	9.9096915
: :	Rad, or sin	90°	10.0000000
:	Sin ∠ C 64° 41′ 6″ or 115° 18′ 54″		9.0561546

INSTRUMENTALLY.

1. Extend the compasses from 35° 41′ (compt. of ∠ B) to 26° 2′ (compt. of A C) on the sines, and this extent will reach, on the same line, from 90° to 48° 48′, the side B C.

2. Extend from 63° 58′ (A C) to 54° 19′ (∠ B) on the tangents, and this extent will reach, on the sines, from 90° to 42° 51′, the ∠ A.

(g) This analogy, by proper substitution, becomes tan A C : tan B : : rad : sin A, which is that used in the instrumental solution.

3. Extend from 63° 58′ (A C) to 54° 19′ (∠ B) on the sines, and this extent will reach, on the same line, from 90° to 64° 41′, the ∠ C.

2. In the quadrantal spherical triangle A B C,

G^n. The side A C 112° 27′
The opp. ∠ B 117° 30′
Required the other parts.
Ans. B C 55° 47′ 39″ or 124° 12′ 21″
∠ A 52° 32′ 9″ or 127° 27′ 51″
∠ C 73° 41′ 17″ or 106° 18′ 33″

3. In the quadrantal spherical triangle A B C,

G^n. The side B C 121° 9′
The opp ∠ A 54° 13′
Required the other parts.
Ans. A C 62° 12′ 35″ or 117° 47′ 25″
∠ B 56° 59′ 24″ or 123° 0′ 35″
∠ C 71° 25′ 30″ or 108° 34′ 30″

CASE II.

When a side and its adjacent angle are given, to find the rest.

1. *To find the other ∠.*

As cot given side : sin adjacent or given ∠ : : rad : tan other ∠.

Which ∠ is of the same kind as its opp. side.

2. *To find the other side.*

As rad : sin giv. side : : cos adj. or giv ∠ : cos other side.

Which side is of the same kind as its opp. ∠.

3. *To find the hypothenusal ∠.*

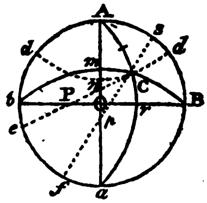
As tan given ∠ : cos adjacent or given side : : rad : cot hypothenusal ∠.

Which hypothenusal ∠ is greater than 90° when the given side and ∠ are like; but less than 90° when they are unlike.

EXAMPLES.

1. In the quadrantal spherical triangle A B C, having given the side A C 63° 58′, and its adjacent angle A 42° 51′, to find the rest.

BY CONSTRUCTION.



1. Describe the circle A *b a* B with the chord of 60°, and draw the diameters A *a*, B *b*, at right angles to each other.

2. Set off A *d*, A *d*, each way, with the chord of A C (63° 58′), and take O *n* equal to the semitangent of the complement of that arc (26° 2′), and O *r* equal to the semitangent of the complement of ∠ A (47° 9′).

3. Through the points *d*, *n*, *d*, and A, *r*, *a*, describe circles cutting each other in C; then if a circle be described through the points *b*, C, B, the triangle A B C will be the one required.

To measure the required parts.

1. Set off the semitangent of the ∠ A (42° 51′) from O to P; also take O *m*, on the same scale, and set off its complement from O to *p*.

2. Through the points C, P, and C, *p*, draw the lines *c e* and *s* C *f*; then B *s*, taken on the scale of chords, gives B C 48° 47′; *e f*, on the same scale, gives ∠ C

25° 19′, (or 154° 41′ 13″); and the complement of the degrees in o *m*, taken on the semitangents, gives ∠ B 54° 18′.

BY CALCULATION.

: Rad, or sin - - - 90° - - - - 10.0000000
: Sin A C - - - - - - 63° 58′ - - - 9.9535369
: : Cos ∠ A - - - - - - 42° 51′ - - 9.8651849
: Cos B C - - - - - - 48° 47′ 43″ 9.8187218

Which side is less than 90°, being like its opp. ∠ A.

(*h*): Cot A C - - - - - - 63° 58′ - - - 9.6888227
: Sin ∠ A - - - - - - 42° 51′ - - - 9.8325609
: : Rad, or sin - - - 90° - - - - 10.0000000
: Tan ∠ B - - - - - 54° 18′ 47″ 10.1437382

Which ∠ is acute, being like its opposite side A C.

: Tan ∠ A - - - - - - 42° 51′ - - 9.9673759
: Cos A C - - - - 63° 58′ - - 9.6423596
: : Rad, or sin - - 90° - - - - 10.0000000
: Cot ∠ C - - - - - - 154° 41′ 13″ 9.6749837

Which ∠ is obtuse, because A C and ∠ A are like.

INSTRUMENTALLY.

1. Extend the compasses from 90° to 63° 58′ (A C) on the sines, and that extent will reach, on the same line, from 47° 49′ (comp^t. of ∠ A) to 41° 13′, the complement of B C.

2. Extend from 90° to 42° 51′ (∠ A) on the sines, and this extent will reach, on the tangents, from 63° 58′ (A C) to 54° 18′, ∠ B.

(*h*) The two last of these analogies, by proper substitutions, become $\left\{\begin{matrix}\text{rad : sin A : : tan A C : tan B}\\ \text{rad : cos A C : : cot A : cot C}\end{matrix}\right\}$ which are those adapted to the use of the instrument.

3. Extend from 90° to 26° 2′ (comp^{t}. of A C) on the sines, and this extent will reach on the tangents, from 47° 9′ (comp^{t}. of ∠ A) to 25° 19′, the comp^{t}. of C.

2. In the quadrantal spherical triangle A B C,

Given { The side A C 112° 27′ / The adj^{t}. ∠ A 120° 15′ } Ans. { B C 117° 44′ 55″ / ∠ B 115° 33′ 46″ / ∠ C 102° 33′ 18″ }
Required the other parts.

3. In the quadrantal spherical triangle A B C,

Given { The side B C 59° 16′ / The adj^{t}. ∠ B 147° 8′ } Ans. { A C 136° 13′ 3″ / ∠ A 42° 23′ 20″ / ∠ C 51° 39′ 27″ }
Required the other parts.

CASE III.

When the hypothenusal angle and either of the other angles are given, to find the rest.

1. *To find the side opposite to the given* ∠.

As sin hyp^{l}. ∠ : rad : : sin given ∠ : sin opposite or required side.

Which side is like its opposite ∠.

2. *To find the side adjacent to the given* ∠.

As rad : cot hyp^{l}. ∠ : tan given ∠ : : cos adjacent or required side.

Which side is greater than 90° when the given ∠^{s} are like; but less than 90° when they are unlike.

3. *To find the remaining* ∠.

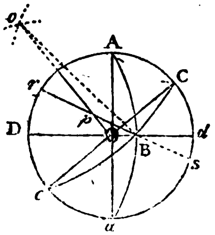
As cos given ∠ : rad : : cos hyp^{l}. ∠ : cos remaining or required ∠.

Which ∠ is obtuse when the given ∠^{s} are like; but acute when they are unlike.

EXAMPLES.

1. In the quadrantal spherical triangle A B C, having the angle A 42° 51′, and the hyp^l. angle C 115° 19′, to find the rest.

BY CONSTRUCTION.



1. Describe the circle A D *a d* with the chord of 60°, and draw the diameters A *a*, D *d*, at right angles to each other.

2. Set off O B equal to the semitangent of the complement of ∠ A (47° 9′) and through the points A, B, *a*, describe a circle.

3. From O, B with the tangent and secant of ∠ C (115° 19′ or 64° 41′) describe arcs cutting each other in O; then from the point O, as a centre, with the radius O P, describe the circle C B *c*, and A B C will be the triangle required.

To measure the required parts.

1. In O *o* set off O *p* equal to the semitangent of ∠ C (115° 19′ or 64° 41′), and through *p*, B, draw the line *r s*, cutting the circle in *r* and *s*.

2. Then *r* D, taken on the scale of chords, gives ∠ B 54° 19′; C *s*, on the same scale, gives C B 48° 47′; and A C is 63° 58′.

BY CALCULATION.

:	Sin ∠ C	115° 19′	9.9561483
:	Rad, or sin	90°	10.0000000
: :	Sin ∠ A	42° 51′	9.8325609
:	Sin B C	48° 47′ 36″	9.8764126

Which side is less than 90°, being like its opp. ∠ A.

:	Rad, or sin	90°	10.0000000
:	Tan ∠ A	42° 51′	9.9673759
: :	Cot ∠ C	115° 19′	9.6749105
:	Cos A C	63° 58′ 17″	9.6422864(i)

Which side is less than 90°, because the ∠ˢ A and C are unlike.

:	Cos ∠ A	42° 51′	9.8651849
:	Rad, or sin	90°	10.0000000
: :	Cos ∠ C	115° 19′	9.6310589
:	Cos ∠ B	54° 19′ 8″	9.7658740

Which ∠ is acute, because the given ∠ˢ are unlike.

INSTRUMENTALLY.

1. Extend the compasses from 47° 9′ (compᵗ. of ∠ A) to 90° on the sines, and this extent will reach, on the same line, from 25° 19′ (compᵗ. of ∠ C) to 35° 41′, the complement of ∠ B.

2. Extend from 64° 41′ (∠ C) to 90° on the sines, and this extent will reach, on the same line, from 42° 51′ (∠ A) to 48° 47′, the side B C.

3. Extend from 64° 41′ (∠ C) to 42° 51′ (∠ A) on the tangents, and this extent will reach, on the sines, from 90° to 26° 2′, the complement of A C.

(i) This analogy may be converted, as before, into tan C : tan A : : rad : cos A C for the instrumental computation.

2. In the quadrantal spherical triangle A B C,

Given { The hyp^l. ∠ C 102° 9′ | The ∠ A - - 115° 17′
Ans. { ∠ B 119° 31′ 29″ | B C 112° 20′ 35″ | A C 117° 6′ 59″
Required the other parts.

3. In the quadrantal spherical triangle A B C,

Given { The hyp^l. ∠ C 102° 5′ | The ∠ B - - 57° 15′
Ans. { ∠ A 67° 14′ 5″ | B C 70° 33′ 36″ | A C 59° 19′ 38″
Required the other parts.

CASE IV.

When the hypothenusal angle and a side are given, to find the rest.

1. *To find the ∠ opposite that side.*

As rad : sin hyp^l. ∠ : : sin given side : sin opposite or required ∠.

Which ∠ is like its opp. side.

2. *To find the ∠ adjacent given side.*

As cot hyp^l. ∠ : rad : : cos given side : tan its adj^t. ∠.

Which ∠ is obtuse when its adjacent side and the hyp^l. ∠ are like; but acute when they are unlike.

3. *To find the other side.*

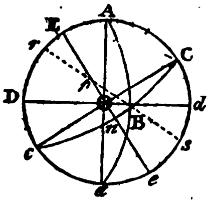
As cot given side : rad : : cos hyp^l. ∠ : cot other side.

Which side is greater than 90°, when the other side and the hyp^l. ∠ are like; but less than 90° when they are unlike.

EXAMPLES.

1. In the quadrantal spherical triangle A B C, having A C 63° 58′, and the hypothenusal ∠ C 115° 19′, to find the rest.

BY CONSTRUCTION.



1. Describe the circle A D *a* *d* with the chord of 60°, and draw the diameters A*a*, D*d*, at right ∠ˢ to each other.

2. Set off A C (63° 58′) from a scale of chords, and draw the diameter C *c*, and another E*e*, at right angles to it.

3. Take o *n* equal to the semitangent of the complement of the ∠ C (25° 19′), and through the points C, *n*, *c*, describe a circle, cutting the diameter D*d* in B.

4. Then if a circle be described through the points A, B, *a*, the triangle A B C will be the one required.

To measure the required parts.

1. Make o *p* equal to the semitangent of the angle C (115° 19′ or 64° 41′), and through *p*, B, draw the line *r s*, cutting the circle in *r* and *s*.

2. Then o B, taken on the semitangents, and subtracted from 90°, gives ∠ A 42° 51′; *r* D, on the chords, gives ∠ B 54° 18′; and C *s*, on the same scale, gives C B 48° 47′.

BY CONSTRUCTION.

:	Rad, or sin - - -	90° - - - -	10.0000000
:	Sin ∠ C - - - - - - -	115° 19′ - -	9.9561483
: :	Sin A C - - - - - - -	63° 58′ - -	9.9535369
:	Sin ∠ B - - - - -	54° 18′ 56″	9.9096852

Which ∠ is acute, being like its opposite side A C.

:	Cot ∠ C	115° 19′	9.6749105
:	Rad, or sin	90°	10.0000000
: :	Cos A C	63° 58′	9.6423596
:	Tan ∠ A	42° 51′ 17″	9.9674491

Which ∠ is acute, because A C and ∠ A are unlike.

:	Cot A C	63° 58′	9.6888227
:	Cos ∠ C	115° 19′	9.6310589
: :	Rad, or sin	90°	10.0000000
:	Cos B C	48° 47′ 57″	9.9422362(k)

Which side is less than 90°, because A C and ∠ C are unlike.

INSTRUMENTALLY.

1. Extend the compasses from 90° to 64° 41′ (∠ C) on the sines, and this extent will reach, on the same line, from 63° 58′ (A C) to 54° 18′, the ∠ B.

2. Extend from 90° to 26° 2′ (compᵗ. of A C) on the sines, and that extent will reach, on the tangents, from 64° 41′ (∠ C) to 42° 51′, the ∠ A.

3. Extend from 90° to 25° 19′ (compᵗ. of ∠ C) on the sines, and this extent will reach, on the tangents, from 63° 58′ (A C) to 41° 13′, the compᵗ. of B C.

2. In the quadrantal spherical triangle A B C,

Given { The side A C 114° 17′
The hypˡ. ∠ C 102° 9′
Required the other parts.

Ans. { B C 115° 0′ 33″
∠ B 116° 59′ 15″
∠ A 117° 37′ 58″

(k) These two analogies are easily converted into the following:

rad : cos A C : : tan C : tan A
rad : cos ∠ C : : tan A C : cot B C,

which are those used for the instrumental computation.

3. In the quadrantal spherical triangle A B C,

Given { The side B C 79° 23′
The hyp^{l}. ∠ C 102° 13′

Required the other parts.

Ans. { A C 41° 32′ 6″
∠ B 40° 23′ 44″
∠ A 73° 52′ 4″

CASE V.

When the two sides are given, to find the rest.

1. *To find either of the other* ∠ˢ.

As sin either side : rad : : cos other side : cos opposite or required ∠.

Which ∠ is like its opposite side.

2. *To find the hypothenusal* /.

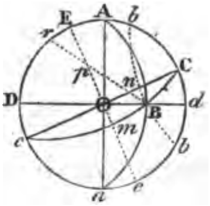
As rad : cot either side : : cot other side : cos hyp^{l}. /.

Which / is obtuse when the given sides are like; but acute when they are unlike.

EXAMPLES.

1. In the quadrantal spherical triangle A B C, having the side A C 63° 58′, and the side B C 48° 48′, to find the rest.

BY CONSTRUCTION.



1. Describe a circle with the chord of 60°, and draw the diameters A *a*, D *d*, at right angles to each other.

2. Take A C equal to the chord of 63° 58′, and set off C *b*, C *b*, each way, with the chord of 48° 48′ (C B).

3. Make O *n* equal to the semitangent of the complement of C B (41° 12′), and through the points *b*, *n*, *b*, describe a circle, cutting D *d* in B.

4. Draw the diameter C *c*, and through the points C, B, *c*, describe a circle, and through A B *a* another; then A B C, will be the triangle required.

To measure the required parts.

1. Draw the diameter E *e* perpendicular to C *c*; and having taken O *m*, in degrees, on the semitangents, set off its complement from O to *p*.

2. Through *p*, B, draw B *p r*, cutting the circle in *r*; then *r* D, on the chords, gives ∠ B 54° 18′; O B, taken on the semitangents, and subtracted from 90°, gives ∠ A 42° 51′; and O *m*, taken on the same line, and then subtracted from 90°, gives ∠ C 64° 42′ or 115° 18′.

BY CALCULATION.

:	Sin B C - - - - -	48° 48′ - - -	9.8764574
:	Rad, or sin - - -	90° - - - -	10.0000000
: :	Cos A C - - - - -	63° 58′ - -	9.6423596
:	Cos ∠ B - - - -	54° 18′ 58″	9.7659022

Which ∠ is acute, being like its opposite side A C.

:	Sin A C - - - - -	63° 58′ - - -	9.9535369
:	Rad, or sin - - -	90° - - - -	10.0000000
: :	Cos B C - - - - -	48° 48′ - - -	9.8186807
:	Cos ∠ A - - - - -	42° 51′ 21″	9.8651438

Which ∠ is acute, being like its opposite side B C.

:	Rad, or sin - - -	90° - - - -	10.0000000
:	Cot A C - - - - -	63° 58′ - - -	9.6888227
: :	Cot B C - - - - -	48° 48′ - - -	9.9422233
:	Cos ∠ C - - - - -	115° 18′ 57″	9.6310460(*l*)

Which ∠ is obtuse, because A C and B C are like.

(*l*) This analogy becomes tan A C : cot B C : : rad : cos C for the instrumental computation.

INSTRUMENTALLY.

1. Extend the compasses from 63° 58′ (A C) to 41° 12′ (compt. of B C) on the tangents, and that extent will reach, on the sines, from 90° to 25° 18′, the complement of ∠ C.

2. Extend from 48° 48′ to 90° on the sines, and that extent will reach, on the same line, from 26° 2′ (compt. of A C) to 35° 42′, the complement of ∠ B.

3. Extend from 63° 58′ (A C) to 90° on the sines, and that extent will reach, on the same line, from 41° 12′ (compt. of B C) to 47° 9′, the complement of ∠ A.

2. In the quadrantal spherical triangle A B C,

Given { The side A C 109° 12′
The side B C 112° 15′ } Ans. { ∠ C 98° 11′ 26″
∠ B 110° 48′ 48″
Required the other parts. ∠ A 113° 38′ 16″ }

3. In the quadrantal spherical triangle A B C,

Given { The side A C 145° 17′
The side B C 81° 12′ } Ans. { ∠ C 77° 5′ 21″
∠ B 146° 16′ 51″
Required the other parts. ∠ A 74° 25′ 4″ }

CASE VI.

When the two angles are given, to find the rest.

1. *To find either of the two sides.*

As tan either given ∠s : rad : : sin other ∠ : cot its adjacent side.

Which side is like its opposite angle.

2. *To find the hypothenusal angle.*

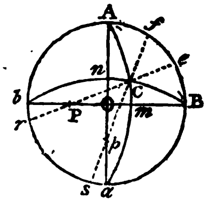
As rad : cos eith. giv. ∠s : : cos other ∠ : cos hypl. ∠.

Which hypl. ∠ is obtuse when the given ∠s are like; but acute when they are unlike.

EXAMPLES.

1. In the quadrantal spherical triangle A B C, having the angle A 42° 51′, and the angle B 54° 19′, to find the rest.

BY CONSTRUCTION.



1. Describe a circle with the chord of 60°, and draw the diameters A *a*, B *b* at right angles to each other.

2. Set off O *m* equal to the semitangent of the complement of ∠ A (47° 9′), and through the points A, *m*, *a* describe a circle.

3. Set off O *n* equal to the semitangent of the complement of ∠ B (35° 41′), and through the points *b*, *n*, B describe a circle, cutting the former in C; then A B C will be the triangle required.

To measure the required parts.

Set off O P, O *p*, equal to the semitangents of the ∠ˢ A and B, and through the point C draw the lines *e r* and *f s*: then *r s*, on the chords, gives ∠ C 115° 19′; A *e*, on the same line, gives A C 63° 58′; and B *f* gives B C 48° 47′.

BY CALCULATION.

:	Tan ∠ A - - - -	42° 51′ - -	9.9673759
:	Rad, or sin - - - -	90° - - - -	10.0000000
: :	Sin ∠ B - - - - -	54° 19′ - -	9.9096915
:	Cot B C - - - - -	48° 47′ 39″	9.9423156

Which side is acute, being like its opposite ∠ A.

:	Tan ∠ B	54° 19′	10.1437958
:	Rad, or sin	90°	10.0000000
: :	Sin ∠ A	42° 51′	9.8325609
:	Cot A C	63° 58′ 11″	9.6887651

Which side is acute, being like its opposite ∠ B.

:	Rad, or sin	90°	10.0000000
:	Cos ∠ A	42° 51′	9.8651849
: :	Cos ∠ B	54° 19′	9.7658957
:	Cos ∠ C	115° 19′ 4″	9.6310806(m)

Which ∠ is obtuse, because the ∠ˢ A and B are like.

INSTRUMENTALLY.

1. Extend the compasses from 54° 19′ (∠ B) to 90° on the sines, and this extent will reach, on the tangents, from 42° 51′ (∠ A) to 48° 47′, the side B C.

2. Extend from 42° 51′ (∠ A) to 90° on the sines, and this extent will reach, on the tangents, from 54° 19′ (∠ B) to 63° 58′, the side A C.

3. Extend from 90° to 47° 9′ (compᵗ. of ∠ A) on the sines, and this extent will reach, on the same line, from 35° 41′ (compᵗ. of ∠ B) to 25° 19′, the complement of ∠ C.

2. In the quadrantal spherical triangle A B C,

Given { The ∠ A 82° 19′
The ∠ B 73° 21′ }
Required the other parts.

Ans. { B C 82° 38′ 7″
A C 73° 29′ 29″
∠ C 92° 11′ 43″ }

(m) The first two of these analogies are convertible into the following ones: { sin B : rad :: tan A : tan B C
sin A : rad :: tan B : tan A C } for the instrumental solution.

3. In the quadrantal spherical triangle A B C,

$$\text{Given}\left\{\begin{matrix}\text{The } \angle \text{ A } 104^\circ\, 19'\\ \text{The } \angle \text{ B } 153^\circ\, 27'\end{matrix}\right. \qquad \text{Ans.}\left\{\begin{matrix}\text{B C } 96^\circ\, 30'\, 27''\\ \text{A C } 152^\circ\, 43'\, 13''\\ \angle \text{ C } 102^\circ\, 46'\, 47''\end{matrix}\right.$$

Required the other parts.

OBLIQUE-ANGLED SPHERICAL TRIANGLES.

The different cases, or varieties, that may happen in the solution of oblique-angled spherical triangles, where any three things are given to find a fourth, are, in all, twelve; but, by restricting them to such as depend upon the same principles, they may be reduced to six. And if a perpendicular be drawn from one of the angles to the opposite side, each of these cases, except the two where the three sides or the three angles are given, may be resolved by means of the rules already proposed for right-angled spherical triangles.

Or, all the cases of oblique-angled spherical triangles may be resolved, without drawing a perpendicular, by means of one or the other of the four following theorems; which are better adapted to practice, and more easily retained in the memory, than the various particulars which must be attended to in the former method, with respect to the falling of the perpendicular, and the species of the different parts of the triangle.

I.

Sin either side : sin its opp. ∠ : : sin any other side : sin its opp. ∠ ; and conversely.

II.

$$\left.\begin{matrix}\text{Sin}\\ \text{or}\\ \text{Cos}\end{matrix}\right\}\tfrac{1}{2}\text{ sum any two sides : }\left.\begin{matrix}\text{sin}\\ \text{or}\\ \text{cos}\end{matrix}\right\}\tfrac{1}{2}\text{ their diff. : : cot } \tfrac{1}{2}\text{ included } \angle \text{ : tan}\left\{\begin{matrix}\tfrac{1}{2}\text{ diff.}\\ \text{or}\\ \tfrac{1}{2}\text{ sum}\end{matrix}\right.\text{ other two } \angle^{\text{s}}.$$

III.

$$\left.\begin{matrix}\text{Sin}\\ \text{or}\\ \text{Cos}\end{matrix}\right\}\tfrac{1}{2}\text{ sum any two }\angle^{s} : \left.\begin{matrix}\text{sin}\\ \text{or}\\ \text{cos}\end{matrix}\right\}\tfrac{1}{2}\text{ their diff. : : tan }\tfrac{1}{2}$$

$$\text{included side : tan}\left\{\begin{matrix}\tfrac{1}{2}\text{ diff.}\\ \text{or}\\ \tfrac{1}{2}\text{ sum}\end{matrix}\right.\text{ other two sides.}$$

IV.

$$:\text{ Rect. sines of any two sides}\left\{\begin{matrix}\text{or of any}\\ \text{two }\angle^{s}\end{matrix}\right\} : \text{rad}^{2}.$$

$$: :\left\{\begin{matrix}\text{sin }\tfrac{1}{2}\text{ sum 3 sides}\times\text{sin diff. this }\tfrac{1}{2}\text{ sum and 3d side}\\ \text{or}\\ \text{cos }\tfrac{1}{2}\text{ sum 3 }\angle^{s}\times\text{cos diff. this }\tfrac{1}{2}\text{ sum and 3d }\angle\end{matrix}\right.$$

$$:\text{ cos}^{2}\ \tfrac{1}{2}\text{ included }\angle\ \{\text{or sin}^{2}\ \tfrac{1}{2}\text{ included side}\}.$$

To which rules it is proper to add, that in the 2d and 3d cases, the terms of the proportion may be taken either directly, inversely, or alternately; and that, in the 4th case, the first four terms of the two analogies, contained in the rule, are to be used for finding an angle, and the four latter for finding a side.

It may here also be observed, that the six cases of oblique-angled spherical triangles, already mentioned, are as follows:

$$\text{Given}\left\{\begin{array}{l}\text{Two sides and an }\angle\text{ opp. to one of them}\\ \text{Two }\angle^{s}\text{ and a side opp. to one of them}\\ \text{Two sides and their included }\angle\\ \text{Two }\angle^{s}\text{ and their included side}\\ \text{The three sides}\\ \text{The three angles}\end{array}\right\}\text{to find the rest.}$$

CASE I.

When two sides and an angle opposite to one of them are given, to find the rest.

1. *To find the other opposite angle.*

As sin side opp. given ∠ : sin that ∠ : : sin other given side : sin its opposite ∠.

Which ∠ is either an acute ∠ or its supplement, according as it makes the greater side opposite the greater angle. And if each of them agree with this rule, the triangle is ambiguous, or admits of two different solutions.

2. *To find the angle contained by the given sides.*

Find the ∠ opp. the other given side, by rule I, and note whether it be ambiguous or not.

Then, sin ½ dif. two given sides : sin ½ their sum : : tan ½ dif. their opp. ∠^{s} : cot ½ inclu^{d}. ∠.

Which ½ ∠ is always acute; and if the angle found by rule I. be ambiguous, the required angle will be ambiguous, otherwise not.

3. *To find the third side.*

Find the angle opp. the other given side, by rule I, and note whether it be ambiguous or not.

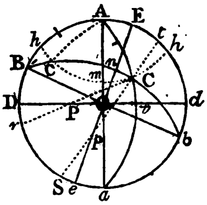
Then, sin ½ dif. these ∠^{s} : sin ½ their sum : : tan ½ dif. given sides : tan ½ remaining side.

Which ½ side is always less than 90°; and if the angle found by rule I. be ambiguous, the required side will be ambiguous, otherwise not.

EXAMPLES.

1. In the oblique-angled spherical triangle A B C, having the side A B 68° 37′, the side A C 62° 40′, and the angle B 51° 28′, to find the rest.

BY CONSTRUCTION.



1. Describe a circle with the chord of 60°, and draw the diameters A *a*, D *d* at right angles to each other.

2. Set off the side A B (68° 37′) from A to B, by the scale of chords, and draw the diameter B *b*, and another E *e*, at right angles to it.

3. Take O *n* equal to the semitangent of the complement of ∠ B (38° 32′), and through the points B, *n*, *b* describe a circle.

4. Set off the side A C (62° 40′), by a scale of chords, from A each way, to *h*, *h*; and make O *m* equal to the semitangent of the complement of A C (27° 20′).

5. Through the points *h*, *m*, *h* describe a circle, cutting the former B *n b* in C; then, if a circle be described through the points A, C, *a*, the triangle A B C, or A B *c*, will be the one required, each having the same data; which shows the case to be ambiguous.

To measure the required parts.

1. Take O *v* in degrees, on the semitangents, and set off its complement from O to P: also take O *p* equal to the semitangent of ∠ B (51° 28′).

2. Through the points C, P and C, *p* draw the lines C *r*, and S *t*, cutting the circle in *r*, S, and *t*; then *r* S, on the chords, gives the ∠ C 55° 4′; O *v*, taken on the line of semitangents, and then added to 90°, gives ∠ A 122° 4′; and B *t*, on the chords, gives B C 105° 45′.

And if the triangle A B *c* had been taken, the angle B A *c* would have been found 8° 44′, angle B A *c* 24° 36′, and B *c* 9° 55′.

BY CALCULATION.

: Sin A C - - - - - 62° 40′ - - - 9.9485842
0.0514158
: Sin ∠ B - - - - - 51° 28′ - - - 9.8933433
: : Sin A B - - - - - 68° 37′ - - - 9.9690252
: Sin ∠ C 55° 4′ 45″, or 124° 55′ 15″ 9.9137843

: Sin $\frac{AB \sim AC}{2}$ - - - 2° 58′ 30″ - - 8.7151692
1.2848308
: Sin $\frac{AB + AC}{2}$ - - - 65° 38′ 30″ - - 9.9595107
: : Tan $\frac{C \sim B}{2}$ - - - - 1° 48′ 22″ - - 8.4987656
: Cot $\frac{\angle A}{2}$ - - - - - 61° 2′ 12″ - - 9.7431071
2
122° 4′ 24″ ∠ A

And had ∠ *c* 124° 55′ 15″ been taken, the result, by the same method, would have given ∠ A, or B A *c*, 8° 44′.

: Sin $\frac{C \sim B}{2}$ - - - - 1° 48′ 22″ - - 8.4985498
1.5014502
: Sin $\frac{C + B}{2}$ - - - - 53° 16′ 22″ - - 9.9038989
: : Tan $\frac{AB \sim AC}{2}$ - - 2° 58′ 30″ - - 8.7857549
: Tan $\frac{BC}{2}$ - - - - 52° 53′ 13″ - - 10.1211040
2
105° 46′ 26″ B C.

And had ∠ *c* 124° 55′ 15″ been taken in this case, the result, by the same method, would have given BC 9° 55′ 44″ (n).

INSTRUMENTALLY.

1. Extend the compasses from 62° 40′ (A C) to 51° 28′ (∠ B) on the sines, and this extent will reach, on the same line, from 68° 37′ (A B) to 55° 4′ ∠ C.

2. Extend from 2° 58′ $\left(\frac{AB \sim AC}{2}\right)$ to 65° 38′ $\left(\frac{AB + AC}{2}\right)$ on the sines, and this extent will reach, on the tangents, from 1° 48′ $\left(\frac{C \sim B}{2}\right)$ to 28° 58′, the compt. of ½ ∠ A.

3. Extend from 1° 48′ $\left(\frac{C \sim B}{2}\right)$ to 53° 16′ $\left(\frac{C + B}{2}\right)$ on the sines, and this extent will reach, on the tangents, from 2° 58′ $\left(\frac{AB \sim AC}{2}\right)$ to 52° 53′, which is ½ side B C.

2. In the oblique-angled spherical triangle A B C,

Given	The side A B 56° 19′ The side A C 114° 12′ The ∠ B - - 121° 50′	Ans.	B C 87° 3′ 52″ ∠ C 50° 48′ 39″ ∠ A 68° 28′ 7″

Required the other parts.

3. In the oblique-angled spherical triangle A B C,

Given	The side A C 62° 27′ The side B C 73° 19′ The ∠ A - - 51° 14′	Ans.	A B 116° 47′ 22″ ∠ B 46° 11′ 32″ ∠ C 133° 23′ 50″

Required the other parts.

(n) This example affords an opportunity of remarking, that the angle first found, or that opposite the other given side, in this case, is always either an acute angle or its supplement; but it is evident, both from the construction and calculation, that this may not be the case with respect to the remaining side, or the remaining angle; for the two values of ∠ A are 122° 4′ 22″ and 8° 44′, and of B C 105° 46′ 26″ and 9° 55′ 44″, which are not supplements of each other.

4. In the oblique-angled spherical triangle A B C,

Given	The side A B 59° 19′	Ans.	A C 43° 18′ 36″
	The side B C 48° 7′		∠ A 59° 50′ 57″
	The ∠ C - 87° 15′		∠ B 52° 48′ 58″

Required the other parts.

CASE II.

When two angles and a side opposite to one of them are given, to find the rest.

1. *To find the other opposite side.*

As sin ∠ opp. given side : sin that side : : sin other given ∠ : sin its opposite side.

Which side is an arc less than 90°, or its supplement, according as it makes the greater side opposite the greater angle.

And if each of them agree with this rule, the triangle is ambiguous, or admits of two different solutions.

2. *To find the side included by the given ∠s.*

Find the side opp. the other given angle, by rule I, and note whether it be ambiguous or not.

Then,

Sin ½ dif. two given ∠s : sin ½ their sum : : tan ½ dif. their opposite sides : tan ½ included side.

Which ½ side is always less than 90°; and if the side found by rule I. be ambiguous, the required side will be ambiguous, otherwise not.

3. *To find the remaining angle.*

Find the side opp. the other given angle, by rule I, and note whether it be ambiguous or not.

Then,

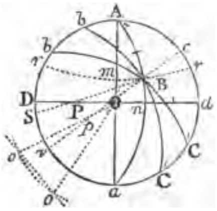
Sin ½ dif. two given sides : sin ½ their sum : : tan ½ dif. their opposite ∠s : cot ½ included ∠.

Which ½ ∠ is always acute; and if the side, found by rule I, be ambiguous, the required angle will be ambiguous, otherwise not.

EXAMPLES.

1. In the oblique-angled spherical triangle A B C, having the angle C 51° 38′, the angle A 59° 27′, and the side A B 64° 5′, to find the rest.

BY CONSTRUCTION.



1. Describe a circle with the chord of 60°, and draw the diameters A *a*, D *d* at right angles to each other.

2. Take O *n* equal to the semitangent of the complement of ∠ A (30° 33′), and through the points A, *n*, *a* describe a circle.

3. Set off A *r*, A *r*, each equal to A B (64° 5′) from a scale of chords, and having made O *m* equal to the semitangent of the complement of A B (25° 55′), describe the circle *r m r*, cutting A B *a* in B.

4. With the tangent of ∠ C (51° 38′), and O as a centre, describe an arc; and with the secant of the same angle, and B as a centre, cross it in *o*.

5. From the centre *o*, with radius O B, describe the

circle *b* B C; and A B C, or A B C′, will be the triangle required, each having the same data; which shows the question to be ambiguous.

To measure the other parts.

1. Set off the semitangent of the compᵗ. of ∠ A (30° 33′) from O to P, and draw B P S; also, with the semitangent of ∠ C (51° 38′), and O as a centre, cross O *o* in *p*, and draw *v p* B C.

2. Then S *v*, taken on the scale of chords, and subtracted from 180°, gives ∠ A B C 132° 24′; *c* C, on the same scale, gives B C 81° 4′; and A C is 122° 6′.

And, if the triangle A B C′ had been taken, the angle A B C′ would have been found 154° 33′, the side B C′ 98° 55′, and A C′ 150° 28′.

BY CALCULATION.

: Sin ∠ C - - - - - - 51° 38′ - - - 9.8943464

0.1056536

: Sin A B - - - - - - 64° 5′ - - - 9.9539677

: : Sin ∠ A - - - - - - 59° 27′ - - - 9.9350969

: Sin B C 81° 4′ 56″ or 98° 55′ 4″ 9.9947182

Where it is to be observed, that, as each of these values of B C, agree with the rule, in making the greater side opposite to the greater ∠, the △, in this case, is ambiguous.

: Sin $\frac{C \sim A}{2}$ - - - - 3° 54′ 30″ - 8.8335321

1.1664679

: Sin $\frac{C + A}{2}$ - - - - 55° 32′ 30″ - 9.9162106

: : Tan $\frac{AB \sim BC}{2}$ - - 8° 29′ 58″ - 9.1744700

: Tan $\frac{AC}{2}$ - - - - - - 61° 3′ 1″ 10.2571485

2

122° 6′ 2″ A C

And had B C′ (98° 55′ 4″) been taken, the result, by the same method, would have given A C′ 150° 28′ 32″.

: Sin $\frac{AB \sim BC}{2}$ - - - - 8° 29′ 58″ - 9.1696739

0.8303261

: Sin $\frac{AB + BC}{2}$ - - - - 72° 34′ 58″ - 9.9796169

: : Tan $\frac{C \sim A}{2}$ - - - - - 3° 54′ 30″ - 8.8345423

: Cot $\frac{\angle B}{2}$ - - - - - - - 66° 12′ 1″ - 9.6444853

2

132° 24′ 2″ ∠ A B C

And had B C′ (98° 55′ 4″) been taken, the result, by the same method, in this case, would have given ∠ A B C′ 154° 33′ 32″ (*o*).

INSTRUMENTALLY.

1. Extend the compasses from 51° 38′ (∠ C) to 64° 5′ (A B) on the sines, and this extent will reach, on the same line, from 59° 27′ (∠ A) to 81° 4′, the side B C.

2. Extend from 3° 54′ $\left(\frac{C \sim A}{2}\right)$ to 55° 32′ $\left(\frac{C + A}{2}\right)$ on the sines, and this extent will reach, on the tangents, from 8° 29′ $\left(\frac{AB \sim BC}{2}\right)$ to 61° 3′, which is ½ A C.

(*o*) From this example it also appears that the side B C, first found, must be either an arc less than 90°, (81° 4′ 56″) or its sup^t. (98° 55′ 4″); but the two values of the side A C (122° 6′ 2″ and 150° 28′ 32″) are both obtuse, as are, also, the two values of ∠ B (132° 24′ 3″ and 154° 33′ 32″).

3. Extend from 8° 29′ $\left(\frac{\text{AB} \sim \text{BC}}{2}\right)$ to 72° 34′ $\left(\frac{\text{AB} + \text{BC}}{2}\right)$ on the sines, and this extent will reach, on the tangents, from 3° 54′ $\left(\frac{\text{C} \sim \text{A}}{2}\right)$ to 23° 48′, the complement of ½ ∠ B.

2. In the oblique-angled spherical triangle A B C,

Given	The ∠ C - - - 49° 7′	Ans.	A B 55° 53′ 21″
	The ∠ A - - - 124° 13′		A C 85° 53′ 27″
	The side B C 115° 6′		∠ B 65° 36′ 58″

Required the other parts.

3. In the oblique-angled spherical triangle A B C,

Gⁿ.	The ∠ C - - 47° 19′	Ans.	B C 63° 13′ 22″ or 116° 46′ 39″
	The ∠ A - - 51° 32′		A C 97° 2′ 46″ or 170° 22′ 32″
	The side A B 56° 57′		∠ B 119° 29′ 34″ or 171° 34′ 6″

Required the other parts.

CASE III.

✓ When two sides and their included angle are given, to find the rest.

3. *To find either of the other two angles.*

As cos ½ sum given sides : cos ½ their diff. : : cot ½ their inclu^d. ∠ : tan ½ sum other two ∠ˢ.

And, As sin ½ sum given sides : sin ½ their diff. : : cot ½ their inclu^d. ∠ : tan ½ diff. other two ∠ˢ.

Then, if the ½ diff. of these two ∠ˢ be added to ½ their sum, it will give the ∠ opp. the greater side; and, if it be subtracted from the ½ sum, it will give the angle opposite the less side.

In which case, ½ the sum of the two ∠ˢ is like ½ the sum of their opposite sides, and ½ their difference is always less than 90°.

2. *To find the remaining side.*

Find the two remaining angles by the first part of the rule. Then,

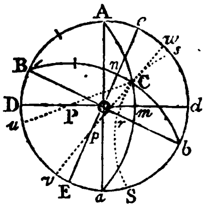
Sin ½ diff. of these ∠s : sin ½ their sum : : tan ½ diff. their opp. sides : tan ½ remaining side.

Which ½ side is always less than 90°.

EXAMPLES.

1. In the oblique-angled spherical triangle A B C, having the side A B 76° 20′, the side B C 119° 17′, and the included angle B 52° 5′, to find the rest.

BY CONSTRUCTION.



1. Describe the circle A D *a d* with the chord of 60°, and draw the diameters A *a*, D *d* at right angles to each other.

2. Set off A B (76° 20′) from a scale of chords, and draw the diameter B *b*, and another E *e*, at right angles to it.

3. Make O *n* equal to the semitangent of the complement of ∠ B (37° 55′), and through the points B, *n*, *b* describe a circle.

4. Set off *b* S, *b s* each equal to the chord of the supplement of B C (60° 43′), and having made O *r* equal to the semitangent of the complement of the same arc, describe the circle S *r s*, cutting the circle B *n b* in C.

5. Through the points A, C, *a* describe a circle; and A B C will be the triangle required.

To measure the required parts.

1. Take O *m*, O *n*, in degrees on the line of semitangents, and set their complements from O to P and *p*; and through the points C, P, C, *p*, draw the lines *u* C, *v w*.

2. Then A *w*, measured on the scale of chords, gives A C 66° 5′; *u v*, on the chords, gives ∠ C 56° 58′; and O *m*, taken on the semitangents, and added to 90°, gives ∠ A 131° 10′.

BY CALCULATION.

:	Cos $\frac{AB+BC}{2}$	97° 48′ 30″	9.1330906
			0.8669094
:	Cos $\frac{AB\sim BC}{2}$	21° 28′ 30″	9.9687525
: :	Cot $\frac{\angle B}{2}$	26° 2′ 30″	10.3110171
:	Tan $\frac{\angle A+\angle C}{2}$	85° 55′ 10″ or 94° 4′ 50″	11.1466790

The latter of which 94° 4′ 50″ must be taken, to make the ½ sum of the ∠s agree with the ½ sum of their opposite sides.

:	Sin $\frac{AB+BC}{2}$	97° 48′ 30″	9.9959545
			0.0040455
:	Sin $\frac{AB\sim BC}{2}$	21° 28′ 30″	9.5635941
: :	Cot $\frac{\angle B}{2}$	26° 2′ 30″	10.3110171
:	Tan $\frac{\angle A\sim\angle C}{2}$	37° 5′ 52″	9.8786567
		94° 4′ 50″	
		131° 10′ 42″ ∠ A	
		56° 58′ 58″ ∠ C	

: Sin $\frac{\angle A \sim \angle C}{2}$ - - 37° 5′ 52″ - - $\underline{9.7804447}$

0.2195553

: Sin $\frac{\angle A + \angle C}{2}$ - - 94° 4′ 50″ - - 9.9988977

: : Tan $\frac{AB \sim BC}{2}$ - - 21° 28′ 30″ - - $\underline{9.5948416}$

: Tan $\frac{AC}{2}$ - - - - 33° 2′ 48″ - - $\underline{9.8132946}$

2

$\underline{\overline{66^\circ\ 5'\ 36''}}$ AC.

INSTRUMENTALLY.

1. Extend the compasses from 7° 48′ (compᵗ. of $\frac{AB + BC}{2}$) to 68° 32′ (compᵗ. of $\frac{AB \sim BC}{2}$) on the sines, and this extent will reach, on the tangents, from 63° 58′ (compᵗ. of ½∠B) to 85° 55′, the ½ sum of ∠ˢ A and C.

2. Extend from 82° 11½′ (suppᵗ. of $\frac{AB + BC}{2}$) to 21° 28′ ($\frac{AB \sim BC}{2}$) on the sines, and the same extent will reach, on the tangents, from 63° 58′ (compᵗ. of ½∠B) to 37° 5′, the ½ diff. of ∠ˢ A and C.

3. Extend from 37° 5′ ($\frac{\angle A \sim \angle C}{2}$) to 94° 4′ ($\frac{\angle A + \angle C}{2}$) on the sines, and this extent will reach, on the tangents, from 21° 28′ ($\frac{AB \sim BC}{2}$) to 33° 2′, the ½ of AC.

2. In the oblique-angled spherical triangle ABC,

$$\text{Given}\begin{cases}\text{The side AB} & 59^\circ\ 13'\\ \text{The side BC} & 81^\circ\ 17'\\ \text{The inc}^{d}.\ \angle B & 125^\circ\ 36'\end{cases}\ \text{Ans.}\begin{cases}\angle A & 62^\circ\ 8'\ 52''\\ \angle C & 50^\circ\ 12'\ 58''\\ AC & 114^\circ\ 37'\ 54''\end{cases}$$

Required the other parts.

3. In the oblique-angled spherical triangle ABC,

Given	The side AB 109° 19′	Ans.	∠ C 108° 44′ 1″
	The side BC 78° 14′		∠ A 79° 14′ 37″
	The inc^{d}. ∠ B 91° 9′		AC 94° 56′

Required the other parts.

CASE IV.

When two angles and their included side are given, to find the rest.

1. *To find either of the other two sides.*

As cos ½ sum given ∠^{s} : cos ½ their diff. : : tan ½ included side : tan ½ sum other two sides.

And,

As sin ½ sum given ∠^{s} : sin ½ their diff. : : tan ½ included side : tan ½ diff. other two sides.

And if ½ the diff. of the sides be added to ½ their sum, it will give the side opposite the greater ∠; and, if it be subtracted from ½ the sum, it will give the side opposite the less ∠.

In which case, ½ the sum of the two sides is like ½ the sum of their opposite ∠^{s}, and ½ their diff. is less than 90°.

2. *To find the remaining angle.*

Find the two remaining sides by the former part of the rule.

Then,

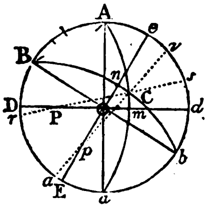
Sin ½ diff. these sides : sin ½ their sum : : tan ½ diff. given ∠^{s} : cot ½ remaining ∠.

Which ½ ∠ is always less than 90°.

EXAMPLES.

1. In the oblique-angled spherical triangle A B C, having the angle A 130° 56′, the angle B 52° 5′, and the included side A B 76° 48′, to find the rest.

BY CONSTRUCTION.



1. Describe the circle A D a d with the chord of 60°, and draw the diameters A a, D d, at right angles to each other.

2. Set off A B (76° 48′) from a scale of chords, and draw the diameter B b, and another E e at right angles to it.

3. Make O n equal to the semitangent of the complement of the ∠ B (37° 55′), and through the points B, n, b describe a circle.

4. Also, make O m equal to the semitangent of 40° 56′ (the excess of ∠ A above 90°); and through the points A, m, a describe a circle, cutting the former in C; and A B C will be the triangle required.

To measure the required parts.

1. Set off the semitangent of 52° 5′ (∠ B) from

O to *p*, and the semitangent of 49° 4′ (sup^t. of ∠ A) from O to P.

2. Through the points P, C and *p*, C draw the lines *r s*, *a v*: then A *s*, on the chords, gives A C 65° 44′; B *v*, on the same line, gives B C 119° 11′; and *r a* is 57° 24′, the ∠ C.

BY CALCULATION.

: $\cos\frac{\angle A + \angle B}{2}$ - -	91° 30′ 30″ - -	8.4203245
		1.5796755
: $\cos\frac{\angle A \sim \angle B}{2}$ - -	39° 25′ 30″ - -	9.8878741
: : $\tan\frac{AB}{2}$ - - - - -	38° 24′ - -	9.8990487
: $\tan\frac{AC + CB}{2}$	87° 32′ 17″ or 92° 27′ 43″	11.3665983

The latter of which 92° 27′ 43″ must be taken, to make the ½ sum of the sides agree with the ½ sum of their opposite angles.

: $\sin\frac{\angle A + \angle B}{2}$ - -	91° 30′ 30″ - -	9.9998495
		0.0001505
: $\sin\frac{\angle A \sim \angle B}{2}$ - -	39° 25′ 30″ - -	9.8028200
: : $\tan\frac{AB}{2}$ - - - - -	38° 24′ - -	9.8990487
: $\tan\frac{AC \sim CB}{2}$ - -	26° 43′ 34″ - -	9.7020192
	92° 27′ 43″	
	119° 11′ 17″ B C	
	65° 44′ 9″ A C.	

: Sin $\frac{AC \sim CB}{2}$ - - - 26° 43′ 34″ - - 9.6529480

0.3470520

: Sin $\frac{AC + CB}{2}$ - - - 92° 27′ 43″ - - 9.9995989

: : Tan $\frac{\angle A \sim \angle B}{2}$ - - 39° 25′ 30″ - - 9.9149459

: Cot $\frac{\angle C}{2}$ - - - - - 28° 42′ 6″ - 10.2615968

2

57° 24′ 12″ ∠ C.

INSTRUMENTALLY.

1. Extend the compasses from 1° 30′ (comp^{t}. of $\frac{A + B}{2}$) to 50° 35′ (comp^{t}. of $\frac{A \sim B}{2}$) on the sines, and this extent will reach, on the tangents, from 38° 24′ ($\frac{AB}{2}$) to 87° 32′, which is $\frac{AC + BC}{2}$.

2. Extend from 88° 30′ (sup^{t}. of $\frac{A + B}{2}$) to 39° 25′ ($\frac{A \sim B}{2}$) on the sines, and this extent will reach, on the tangents, from 38° 24′ ($\frac{AB}{2}$) to 26° 43′, which is $\frac{AC \sim CB}{2}$.

3. Extend from 26° 43′ ($\frac{AC \sim CB}{2}$) to 2° 27′ (comp^{t}. of $\frac{AC + CB}{2}$) on the sines, and the same extent will reach, on the tangents, from 39° 25′ ($\frac{\angle A \sim \angle B}{2}$) to 61° 18′, the complement of $\frac{\angle C}{2}$.

2. In the oblique-angled spherical triangle ABC,

G^{n}.	The ∠ A - - - 71° 18′	Ans.	BC 80° 15′ 27″
	The ∠ B - - - 130° 15′		AC 127° 25′ 31″
	The inc^{d}. side AB 82° 9′		∠ C 72° 11′ 16″

Required the other parts.

3. In the oblique-angled spherical triangle A B C,

$$\text{G}^{n}.\begin{cases}\text{The } \angle \text{A} \;\text{- - -}\; 110^\circ\, 16' \\ \text{The } \angle \text{C} \;\text{- - -}\; 56^\circ\, 22' \\ \text{The inc}^{d}.\text{side AC } 95^\circ\, 36' \end{cases} \text{Ans.}\begin{cases}\angle \text{B} \;\; 83^\circ\, 21'\, 32'' \\ \text{AB} \;\; 56^\circ\, 32'\; 6'' \\ \text{BC} \;\; 109^\circ\, 57'\, 42'' \end{cases}$$

Required the other parts.

CASE V.

When the three sides are given, to find either of the angles.

As rect. sines of any two sides ; sin ½ sum of the three sides × sin diff. this ½ sum and 3d side : : rad^2 : cos^2 ½ their included ∠.

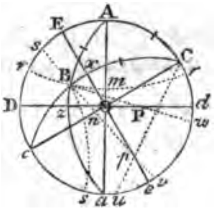
Which ½ ∠ is always acute.

Or the same theorem may be expressed logarithmically thus:—Add together the logarithmic sines of the two sides about the required ∠, and find the complement of their sum, by subtracting the index from 19, and the other figures from 9, as usual. Then, to this complement add the log sine of ½ the sum of the three sides, and the log sine of the diff. of this ½ sum and the third side, and the result, divided by 2, will give the log. cos. of ½ the ∠ sought.

EXAMPLES.

1. In the oblique-angled spherical triangle A B C, having A B 79° 56′, B C 119° 36′, and C A 64° 5′, to find the angles.

BY CONSTRUCTION.



1. Describe a circle with the chord of 60°, and draw the diameters A *a*, D *d* at right angles to each other.

2. Set off A C (64° 5′) from a scale of chords, and draw the diameter C *c*, and another E *e* at right angles to it.

3. Set off the side A B (79° 56′) by a scale of chords, each way from A to *r*; also make O *m* equal to the semitangent of the complement of A B (10° 4′); and through the points *r*, *m*, *r* describe a circle.

4. Set off the supplement of C B (60° 24′) by a scale of chords, each way from *c* to *s*; also make O *n* equal to the semitangent of the complement of C B (29° 36′), and through the points *s*, *n*, *s* describe a circle, cutting the former in B.

5. Through the point B draw the circles A B *a*, and C B *c*, cutting D *d* in *z*, and E *e* in *x*, and A B C will be the triangle required.

To measure the angles.

1. Take O *x*, in degrees, on the semitangents, and set off its complement from O to *p*; also take O *z*, in the same way, and set off its complement from O to P.

2. Through the points C, *p*, draw the line C *p u*; and through B, *p* and B, P draw B *v* and B *w*.

3. Then *v w*, on the chords, gives ∠ B 52° 18′; C *u*, on the same line, gives ∠ C 60° 1′; and O *z*, taken on the line of semitangents, gives 40° 5′ for the excess of ∠ A above 90°.

BY CALCULATION.

AB - - - - 79° 56′
BC - - - - 119° 36′
CA - - - - 64° 5′
2|263° 37′
131° 48′ 30″ - ½ sum
64° 5′ - AC
67° 43′ 30″ - ½ sum — AC.

Sin AB - - - - - - - 79° 56′ - - - 9.9932621
Sin BC - - - - - - - 119° 36′ - - - 9.9392671
19.9325292
0.0674708
Sin ½ sum - - - - - - 131° 48′ 30″ - 9.8723772
Sin (½ sum — AC) - 67° 43′ 30″ - 9.9663179
2|19.9061659
Cos ½ ∠ B - - - - 26° 9′ 20″ - 9.9530829
2
52° 18′ 40″ ∠ B.

131° 48′ 30″ - - ½ sum
79° 56′ - - AB
51° 52′ 30″ - - ½ sum — AB.

Sin BC - - - - - - - 119° 36′ - - - 9.9392671
Sin AC - - - - - - 64° 5′ - - - 9.9539677
9.8932348
0.1067652
Sin ½ sum - - - - - 131° 48′ 30″ - 9.8723772
Sin (½ sum — AB) - 51° 52′ 30″ - 9.8957902
2|19.8749326
Cos ½ ∠ C - - - - - 30° 0′ 53″ - 9.9374663
2
60° 1′ 45″ ∠ C.

131° 48′ 30″ - - ½ sum
119° 36′ - - BC
12° 12′ 30″ - - ½ sum — BC

Sin AC - - - - - - 64° 5′ - - - - 9.9539677
Sin AB - - - - - - 79° 56′ - - - - 9.9932621
9.9472298
0.0527702
Sin ½ sum - - - - 131° 48′ 30″ - - 9.8723772
Sin (½ sum — BC) 12° 12′ 30″ - - 9.3252425
2|19.2503899
Cos ½ ∠ A - - - 65° 2′ 47″ - - 9.6251949
2
130° 5′ 34″ ∠ A.

2. In the oblique-angled spherical triangle ABC,

G^{n}.	The side AB	59° 12′	Ans.	∠ A	62° 39′ 42″
	The side BC	81° 17′		∠ B	124° 50′ 50″
	The side AC	114° 3′		∠ C	50° 31′ 42″

Required the ∠^{s}.

3. In the oblique-angled spherical triangle ABC,

G^{n}.	The side AB	112° 19′	Ans.	∠ A	62° 29′ 48″
	The side BC	85° 16′		∠ B	42° 55′ 54″
	The side AC	49° 56′		∠ C	124° 34′ 40″

Required the ∠^{s}.

4. In the oblique-angled spherical triangle ABC,

Given	The side AB	73° 13′	Ans.	∠ A	44° 18′
	The side BC	62° 42′		∠ B	136° 40′
	The side AC	119° 5′		∠ C	48° 48′

Required the ∠^{s}.

CASE VI.

When the three angles are given, to find either of the sides.

As rect. sines of any two $\angle^s$: cos ½ sum of the three $\angle^s \times$ cos diff. this ½ sum and 3d $\angle$: : rad^2 : sin^2 ½ included side.

Which ½ side is always less than 90°.

Or the same theorem may be expressed logarithmically thus:

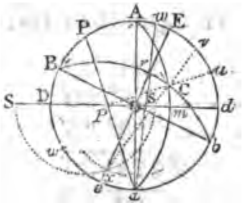
Add together the logarithmic sines of the two $\angle^s$ adjacent to the required side, and find the complement of their sum by subtracting the index from 19, and the rest of the figures from 9, as usual.

Then to this complement add the log cosine of ½ the sum of the three $\angle^s$, and the log cosine of the diff. of this ½ sum and the 3d $\angle$, and the result, divided by 2, will give the log sine of ½ the side sought.

EXAMPLES.

1. In the oblique-angled spherical triangle A B C, having the $\angle$ A 131° 35′, the $\angle$ B 63° 30′, and the $\angle$ C 59° 25′, to find the sides.

BY CONSTRUCTION.



1. Describe the circle A D a d with the chord of 60°, and draw the diameters A a, D d, at right angles to each other.

2. Make O m equal to the semitangent of 41° 35′ (the excess of ∠ A above 90°) and through the points A, m, a describe a circle.

3. Set off O p equal to the semitangent of the complement of the same arc (48° 25′), and draw $a\,p$ P; from which point P set off P w, P w, each equal to the chord of ∠ C (59° 25′).

4. Draw $a\,w'$, $a\,w$, cutting the diameter D d in S, s, and upon S s describe a semicircle: also, from the point O, with the semitangent of ∠ B (63° 30′) as radius, describe an arc cutting the former in x.

5. Through the points x, O, draw the diameter E e, on which take O r equal to the semitangent of the comp^{t}. of ∠ B (26° 30′).

6. Make B b perpendicular to E e, and through the points B, r, b describe a circle, cutting the circle A $m\,a$ in C; and A B C will be the triangle required.

To measure the sides.

Draw $p\,u$ through the points p C, and $x\,v$ through x, C; then A v, taken on the chords, gives A C 80° 17′; B u, on the same line, gives B C 124° 31′; and A B on the same line, is 71° 28′.

BY CALCULATION.

∠A - - - - 131° 35′
∠B - - - - 63° 30′
∠C - - - - 59° 25′
2|254° 30′
127° 15′ - - ½ sum
59° 25′ - - ∠C
67° 50′ - - (½ sum—C).

Sin ∠ A	131° 35′	9.8738965
Sin ∠ B	63° 30′	9.9517912
		9.8256877
		0.1743123
Cos ½ sum	127° 15′	9.7819664
Cos (½ sum ∼ C)	67° 50′	9.5766892
		2|19.5329679
Sin ½ AB	35° 44′ 20″	9.7664839
	2	
	71° 28′ 40″ AB.	

127° 15′ . . . . ½ sum
63° 30′ . . . . ∠ B
63° 45′ . . . . (½ sum ∼ B).

Sin ∠ A	131° 35′	9.8738965
Sin ∠ C	59° 25′	9.9349477
		19.8088442
		0.1911558
Cos ½ sum	127° 15′	9.7819664
Cos (½ sum ∼ B)	63° 45′	9.6457058
		2|19.6188280
Sin ½ AC	40° 8′ 58″	9.8094140
	2	
	80° 17′ 56″ AC.	

131° 35′ . . . ∠ A
127° 15′ . . . ½ sum
4° 20′ . . . (½ sum ∼ A).

Sin ∠ B	63° 30′	9.9517912
Sin ∠ C	59° 25′	9.9349477
		9.8867389
		0.1132611
Cos ½ sum	127° 15′	9.7819664
Cos (½ sum ∼ A)	4° 20′	9.9987567
		2|19.8939842
Sin ½ BC	62° 15′ 49″	9.9469921
	2	
	124° 31′ 39″ BC.	

2. In the oblique-angled spherical triangle ABC,

Given	∠A 71° 42′	Ans.	AB 52° 50′ 44″
	∠B 125° 37′		BC 95° 56′ 10″
	∠C 49° 32′		AC 121° 36′ 31″

Required the sides.

3. In the oblique-angled spherical triangle ABC,

Given	∠A 49° 16′	Ans.	AB 118° 38′
	∠B 63° 17′		BC 57° 45′ 2″
	∠C 128° 9′		AC 85° 31′ 32″

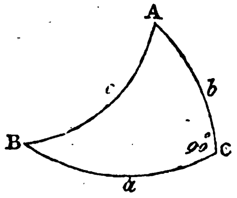
Required the sides.

4. In the oblique-angled spherical triangle ABC,

Given	∠A 121° 21′	Ans.	AB 51° 12′ 21″
	∠B 34° 33′		AC 40° 26′ 45″
	∠C 42° 57′		BC 77° 39′ 31″

Required the sides.

SOLUTIONS OF ALL THE CASES OF RIGHT-ANGLED SPHERICAL TRIANGLES.



I. Given the hypothenuse and either of the oblique angles, to find the other oblique angle.

RULE.

As rad : cos hyp : : tan given ∠ : cot rem^{g}. or req^{d}. ∠.

Which ∠ is acute when the hyp. and given ∠ are like; but obtuse (or the supplement of the former) when they are unlike.

Where it is to be observed, that this affection, as well as all the rest in the following tables, except in the ambiguous cases, may be readily deduced from the algebraic formulæ, by attending to the signs of the quantities which compose them (p).

Or,

$$\text{Cot A} = \frac{\cos c \tan \text{B}}{r}; \ \cot \text{B} = \frac{\cos c \tan \text{A}}{r}.$$

$\text{L} \cot \text{A} = \text{L} \cos c + \text{L} \tan \text{B} - 10$; $\text{L} \cot \text{B} = \text{L} \cos c + \text{L} \tan \text{A} - 10$.

Note. In this, and the two following cases, the hyp. and given ∠ may be each of any magnitude under 180°; but, if either of them be 90°, the required ∠ or leg will be 90°, or else indeterminate.

When the hyp. is = given ∠, cot A will be = sin c, or sin B, and cot B = sin c, or sin A; and when the sum of the hyp. and given ∠ = 180°, cot A will be = − sin c, or sin B; and cot B = − sin c, or sin A.

II. Given the hypothenuse and either of the oblique angles, to find the leg adjacent to that angle.

RULE.

As rad : tan hyp : : cos giv. ∠ : tan adj^{t}. or req^{d}. leg.

Which leg is less than 90° when the hyp. and given ∠ are like; but greater than 90° (or the supplement of the former) when they are unlike.

(p) When a side or angle near 90°, in these or any of the following cases, is to be determined by its tangent, subtract the log tangent from 20, and it will give the log cotangent of the same arc; which can be found, by inspection, to seconds, in the first part of the common tables; and if it should come out in a cotangent, the log tangent may be found, in like manner, by subtracting the log cotangent from 20. See the observations made on this subject in page 27.

Or,

$$\text{Tan } a = \frac{\tan c \cos \text{B}}{r};\ \tan b = \frac{\tan c \cos \text{A}}{r}.$$

L tan $a =$ L tan $c +$ L cos B — 10; L tan $b =$ L tan $c +$ L cos A — 10.

When the hyp. is = given ∠, tan a will be = sin c, or sin B; and tan b = sin c, or sin A.

And when the sum of the hyp. and given ∠ = 180°, tan a will be = — sin c, or sin B; and tan b = — sin c, or sin A.

III. Given the hypothenuse and either of the oblique angles, to find the leg opposite to that angle.

RULE I.

As rad : sin hyp : : sin giv. ∠ : sin opp. or req^{d}. leg.

Which leg is less than 90° when the opp. or given ∠ is acute; but greater than 90° (or the supplement of the former) when it is obtuse.

Or,

$$\text{Sin } a = \frac{\sin c \sin \text{A}}{r};\ \sin b = \frac{\sin c \sin \text{B}}{r}.$$

L sin $a =$ L sin $c +$ L sin A — 10; L sin $b =$ L sin $c +$ L sin B — 10.

RULE II.

Find the other oblique angle by case I; then, by means of the hyp. and this angle, find the leg adjacent to it by case II; which will be the leg required, or that opposite the given angle.

When the hyp. is = given ∠, or the sum of the hyp. and given ∠ = 180°; sin a will be $= \frac{\sin^2 c}{r}$, or $\frac{1}{2}$ vers $2c$.

IV. Given the hypothenuse and either of the legs, to find the angle adjacent to that leg.

RULE I.

As rad : cot hyp :: tan giv. leg : cos adj^t. or req^d. ∠.

Which ∠ is acute when the hyp. and given leg are like; but obtuse (or the supplement of the former) when they are unlike.

Or,

$$\text{Cos A} = \frac{\tan b \cot c}{r}; \ \cos \text{B} = \frac{\tan a \cot c}{r}.$$

$\text{L} \cos \text{A} = \text{L} \tan b + \text{L} \cot c - 10$; $\text{L} \cos \text{B} = \text{L} \tan a + \text{L} \cot c - 10$.

RULE II.

$$\text{Tan } \tfrac{1}{2} \text{A} = r \sqrt{\frac{\sin (c-b)}{\sin (c+b)}}.$$

$$\text{L} \tan \tfrac{1}{2} \text{A} = \frac{\text{C L} \sin (c+b) + \text{L} \sin (c-b) + 10}{2}.$$

The tan of ½ B may also be found by the same formula, using the leg a instead of b: and in each of these cases the ½ ∠ is always acute.

Note. In this, and the two following cases, when the given leg is less than the hypothenuse, their sum is less than 180°; and when it is greater than the hypothenuse, their sum is greater than 180°. If the hyp. be equal to the leg, the triangle is indeterminate.

If the hyp. be = the leg, or the sum of the hyp. and leg = 180°, the △ is indeterminate.

V. Given the hypothenuse and either of the legs, to find the angle opposite to that leg.

RULE I.

As sin hyp : rad : : sin given leg : sin opp. or req^d. ∠.

Which ∠ is acute when the opp. or given leg is less than 90°; but obtuse (or the supplement of the former) when it is greater than 90°.

Or,

$$\text{Sin A} = \frac{r \sin a}{\sin c}; \ \sin \text{B} = \frac{r \sin b}{\sin c}.$$

$\text{L} \sin \text{A} = \text{€L} \sin c + \text{L} \sin a$; $\text{L} \sin \text{B} = \text{€L} \sin c + \text{L} \sin b$.

RULE II.

$$\text{Tan}\,(45° - \tfrac{1}{2}\,\text{A}) = \pm\sqrt{\tan \tfrac{1}{2}\,(c - a) \cot \tfrac{1}{2}\,(c + a)}$$

$$\text{L} \tan\,(45° \sim \tfrac{1}{2}\,\text{A}) = \frac{\text{L} \tan \frac{1}{2}\,(c \sim a) + \text{L} \cot \frac{1}{2}\,(c + a)}{2}.$$

The tan of $(45° - \frac{1}{2}\,\text{B})$ may also be determined by the same form, using b instead of a: and the whole ∠^{s} A or B, which are found by subtracting $2\,(45° \sim \frac{1}{2}\,\text{A})$, or $2\,(45° \sim \frac{1}{2}\,\text{B})$ from 90°, will be acute or obtuse, according to the rule given above.

VI. Given the hypothenuse and either of the legs, to find the other leg.

RULE I.

As cos giv. leg : rad : : cos hyp : cos rem^g. or req^d. leg.

Which leg is less than 90° when the hyp. and given leg are like; but greater than 90° (or the supplement of the former) when they are unlike.

Or,

$$\text{Cos}\ a = \frac{r \cos c}{\cos b}; \ \cos b = \frac{r \cos c}{\cos a}.$$

$\text{L} \cos a = \text{€L} \cos b + \text{L} \cos c$; $\text{L} \cos b = \text{€L} \cos a + \text{L} \cos c$.

RULE II.

$$\text{Tan } \tfrac{1}{2} a = \sqrt{\tan \tfrac{1}{2} (c + b) \tan \tfrac{1}{2} (c - b)}$$

$$\text{L} \tan \tfrac{1}{2} a = \frac{\text{L} \tan \frac{1}{2} (c + b) + \text{L} \tan \frac{1}{2} (c \sim b)}{2}$$

The tangent of $\frac{1}{2} b$ may also be found by the same form, using the leg a instead of b: and in each of these cases the $\frac{1}{2}$ leg is always less than 90°.

VII. Given either of the legs and its adjacent angle, to find the hypothenuse.

RULE.

As cos given ∠ : tan adjacent or given leg : : rad : tan hyp.

Which hyp. is less than 90° when the given leg and angle are like; but greater than 90° (or the supplement of the former) when they are unlike.

Or,

$$\text{Tan } c = \frac{r \tan a}{\cos \text{B}} = \frac{r \tan b}{\cos \text{A}}.$$

$$\text{L} \tan c = \text{€L} \cos \text{B} + \text{L} \tan a = \text{€L} \cos \text{A} + \text{L}\tan b.$$

Note. In this, and the two following cases, the given leg and ∠ may be each of any magnitude under 180°.

When $a = \text{B}$, $\tan c = \frac{r \tan a}{\cos a}$; and if $a + \text{B} = 180°$, $\tan c = -\frac{r \tan a}{\cos a}$.

VIII. Given either of the legs and its adjacent angle, to find the other leg.

RULE.

As rad : sin given leg : : tan adjacent or given ∠ : tan opposite or required leg.

Which leg is less than 90° when the opp. or given ∠ is acute; but greater than 90° (or the supplement of the former) when it is obtuse.

Or,

$$\text{Tan } a = \frac{\sin b \tan \text{A}}{r}; \ \tan b = \frac{\sin a \tan \text{B}}{r}.$$

$\text{L} \tan a = \text{L} \sin b + \text{L} \tan \text{A} - 10$; $\text{L} \tan b = \text{L} \sin a + \text{L} \tan \text{B} - 10$.

When $b = \text{A}$, $\tan a = \frac{\sin b \tan b}{r}$; and if $b + \text{A} = 180°$, $\tan a = -\frac{\sin b \tan b}{r}$.

IX. Given either of the legs and its adjacent angle, to find the other angle.

RULE I.

As rad : sin given ∠ : : cos adjacent or given leg : cos opposite or required ∠.

Which ∠ is acute when the opp. or given leg is less than 90°; but obtuse (or the supplement of the former) when it is greater than 90°.

Or,

$$\text{Cos A} = \frac{\cos a \sin \text{B}}{r}; \ \cos \text{B} = \frac{\cos b \sin \text{A}}{r}.$$

$\text{L} \cos \text{A} = \text{L} \cos a + \text{L} \sin \text{B} - 10$; $\text{L} \cos \text{B} = \text{L} \cos b + \text{L} \sin \text{A} - 10$.

RULE II.

Find the hyp. by case VII; then, by means of the hyp. and given ∠, find the other ∠ by case I.

When $a = \text{B}$, $\cos \text{A} = \frac{1}{2} \sin 2a$; and if $a + \text{B} = 180°$, $\cos \text{A} = -\frac{1}{2} \sin 2a$.

X. Given the two legs, to find either of the oblique ∠s.

RULE.

As sin either leg : rad : : tan other leg : tan opp. or required ∠.

Which ∠ is acute when its opp. leg is less than 90°; but obtuse (or the supplement of the former) when it is greater than 90°.

Or,

$$\text{Tan A} = \frac{r \tan a}{\sin b};\ \tan \text{B} = \frac{r \tan b}{\sin a}.$$

$\text{L} \tan \text{A} = \text{€L} \sin b + \text{L} \tan a$; $\text{L} \tan \text{B} = \text{€L} \sin a + \text{L} \tan b$.

Note. In this, and the following case, the two given legs may be each of any magnitude under 180°.

When $a = b$, $\tan \text{A} = \sec a$, or $\frac{r^2}{\cos a}$; and if $a + b = 180°$, $\tan \text{A} = -\sec \text{A}$, or $-\frac{r^2}{\cos a}$.

XI. Given the two legs, to find the hypothenuse.

RULE I.

As rad : cos either leg : : cos other leg : cos hyp.

Which hyp. is less than 90° when the legs are like; but greater than 90° (or the supplement of the former arc) when they are unlike.

Or,

$$\text{Cos } c = \frac{\cos a \cos b}{r}.$$

$$\text{L} \cos c = \text{L} \cos a + \text{L} \cos b - 10.$$

RULE II.

Find either of the oblique ∠s by case x; then, by means of this angle and its adjacent leg, find the hypothenuse by case VII.

When $a = b$, $\cos c = \frac{\cos^2 a}{r}$; and if $a + b = 180°$, $\cos c = -\frac{\cos^2 a}{r}$.

XII. Given the two oblique angles, to find the hypothenuse.

RULE I.

As rad : cot of eith. of giv. $\angle^s$: : cot other $\angle$: cos hyp.

Which hyp. is less than 90° when the $\angle^s$ are like; but greater than 90° (or the supplement of the former arc) when they are unlike.

Or,

$$\text{Cos } c = \frac{\cot \text{A} \cot \text{B}}{r}.$$

$$\text{L} \cos c = \text{L} \cot \text{A} + \text{L} \cot \text{B} - 10.$$

RULE II.

$$\text{Tan } \tfrac{1}{2} c = r \sqrt{\frac{-\cos(\text{A} + \text{B})}{\cos(\text{A} \sim \text{B})}}.$$

$$\text{L} \tan \tfrac{1}{2} c = \frac{\text{€}\,\text{L} \cos(\text{A} \sim \text{B}) + \text{L} \cos(\text{A} + \text{B}) + 10}{2}.$$

Where $\frac{1}{2} c$ is always less than 90°.

Note. In this, and the following case, the two oblique angles must be so taken, that their difference shall be less than 90°, and their sum between 90° and 270°.

When $\text{A} = \text{B}$, $\cos c = \frac{\cot^2 \text{A}}{r}$, and if $\text{A} + \text{B} = 180°$, $\cos c = -\frac{\cot^2 \text{A}}{r}$.

XIII. Given the two oblique angles, to find either of the legs.

RULE I.

As sin of either of given ∠ : rad : : cos other ∠ : cos opposite or required leg.

Which leg is less than 90° when its opp. ∠ is acute; but greater than 90° (or the supplement of the former arc) when it is obtuse.

Or,

$$\text{Cos } a = \frac{r \cos \mathrm{A}}{\sin \mathrm{B}}; \ \cos b = \frac{r \cos \mathrm{B}}{\sin \mathrm{A}}.$$

$\mathrm{L} \cos a = € \mathrm{L} \sin \mathrm{B} + \mathrm{L} \cos \mathrm{A}$; $\mathrm{L} \cos b = € \mathrm{L} \sin \mathrm{A} + \mathrm{L} \cos \mathrm{B}$.

RULE II.

$$\text{Tan } \tfrac{1}{2} \mathrm{A} = \sqrt{\tan \left(\frac{\mathrm{B}+\mathrm{A}}{2} - 45^\circ\right) \cot \left(\frac{\mathrm{B}-\mathrm{A}}{2} + 45^\circ\right)}$$

$$\mathrm{L} \tan \tfrac{1}{2} a = \frac{\mathrm{L} \tan \left(\frac{\mathrm{B}+\mathrm{A}}{2} - 45^\circ\right) + \mathrm{L} \cot \left(\frac{\mathrm{B}-\mathrm{A}}{2} + 45^\circ\right)}{2}$$

The tan of $\frac{1}{2} b$ may also be found by the same form, putting A in the place of B, and B in that of A: and in each of these cases the $\frac{1}{2}$ leg is always less than 90°.

When $\mathrm{A} = \mathrm{B}$, $\cos a = \cot \mathrm{A}$; and if $\mathrm{A} + \mathrm{B} = 180^\circ$, $\cos \mathrm{A} = - \cot \mathrm{A}$.

XIV. Given either of the legs and its opposite angle, to find the other leg.

RULE I.

As rad : cot given ∠ : : tan of opp. or given leg : sin remaining or required leg.

Which leg is ambiguous: that is, it may be either an arc less than 90°, or its supplement.

Or,

$$\text{Sin } a = \frac{\tan b \cot \text{B}}{r};\ \sin b = \frac{\tan a \cot \text{A}}{r}.$$

$\text{L} \sin a = \text{L} \tan b + \text{L} \cot \text{B} - 10$; $\text{L} \sin b = \text{L} \tan a + \text{L} \cot \text{A} - 10$.

RULE II.

$$\text{Tan } (45^\circ - \tfrac{1}{2} a) = r \sqrt{\frac{\sin (\text{B} - b)}{\sin (\text{B} + b)}}$$

$$\text{L} \tan (45^\circ \sim \tfrac{1}{2} a) = \frac{\text{C L} \sin (\text{B} + b) + \text{L} \sin (\text{B} \sim b) + 10}{2}.$$

The tangent of $(45^\circ - \frac{1}{2} b)$ may also be found by the same form, using A and a instead of B and b: and a or b, which are found by subtracting $2 (45^\circ \sim \frac{1}{2} a)$, or $2 (45^\circ \sim \frac{1}{2} b)$ from 90°, is subject to the same ambiguity as in rule I.

Note. In this, and the two following cases, if the given leg be less than its opposite ∠, their sum will be less than 180°; and if it be greater than its opposite ∠, their sum will be greater than 180°.

Also, if the leg be = its opposite ∠, or their sum be = 180°, the other leg, its opposite ∠, and the hyp. will be each 90°, or else indeterminate.

XV. Given either of the legs and its opposite angle, to find the other angle.

RULE I.

As cos given leg : cos opp. or given ∠ : : rad : sin remaining or required ∠.

Which ∠ is ambiguous; that is, it may be either an acute ∠ or its supplement.

Or,

$$\text{Sin } A = \frac{r \cos B}{\cos b};\ \sin B = \frac{r \cos A}{\cos a}.$$

$\text{L} \sin A = \text{€} \, \text{L} \cos b + \text{L} \cos B$; $\text{L} \sin B = \text{€} \, \text{L} \cos a + \text{L} \cos A$.

RULE II.

$$\text{Tan } (45^\circ - \tfrac{1}{2} A) = \sqrt{\tan \tfrac{1}{2} (B + b) \tan \tfrac{1}{2} (B - b)}$$

$$\text{L} \tan (45^\circ \sim \tfrac{1}{2} A) = \frac{\text{L} \tan \frac{1}{2} (B + b) + \text{L} \tan \frac{1}{2} (B \sim b)}{2}.$$

The tangent of $(45^\circ - \frac{1}{2} B)$ may also be found by the same form, using A and a instead of B and b: and A or B, which is found by subtracting $2 (45^\circ \sim \frac{1}{2} A)$, or $2 (45^\circ \sim \frac{1}{2} B)$ from 90°, is subject to the same ambiguity as in rule I.

XVI. Given either of the legs and its opposite angle, to find the hypothenuse.

RULE I.

As sin given ∠ : sin opp. or given leg : : rad : sin hyp.

Which hyp. is ambiguous; that is, it may be either an arc less than 90°, or its supplement.

Or,

$$\text{Sin } c = \frac{r \sin a}{\sin A} = \frac{r \sin b}{\sin B}.$$

$$\text{L} \sin c = \text{€} \, \text{L} \sin A + \text{L} \sin a = \text{€} \, \text{L} \sin B + \text{L} \sin b.$$

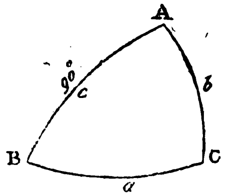
RULE II.

$$\text{Tan } (45^\circ - \tfrac{1}{2} c) = \sqrt{\cot \tfrac{1}{2} (A + a) \tan \tfrac{1}{2} (A - a)}$$

$$\text{L} \tan (45^\circ \sim \tfrac{1}{2} c) = \frac{\text{L} \cot \frac{1}{2} (A + a) + \text{L} \tan \frac{1}{2} (A \sim a)}{2}.$$

The same form may also be applied to the other side and its opposite ∠, using B and b instead of A and a: and the hyp. c, which is found by subtracting $2\ (45^\circ \sim \frac{1}{2}\ c)$ from 90°, will be subject to the same ambiguity as if found by rule I.

SOLUTIONS OF ALL THE CASES OF QUADRANTAL SPHERICAL TRIANGLES.



I. Given the hypothenusal angle and either of the sides, to find the other side.

RULE.

As rad : cos hyp^l. ∠ : : tan given side : cot remaining or required side.

Which side is less than 90° when the given side and hyp^l. ∠ are unlike; but greater than 90° (or the supplement of the former) when they are like.

Or,

$$\text{Cot}\ a = -\frac{\tan b \cos \text{C}}{r};\ \cot b = -\frac{\tan a \cos \text{C}}{r}.$$

$\text{L} \cot a = \text{L} \tan b + \text{L} \cos \text{C} - 10;\ \text{L} \cot b = \text{L} \tan a + \text{L} \cos \text{C} - 10.$

Note. In this, and the two following cases, the given side and hyp^l. ∠ may be each of any magnitude under

$180°$; but, if either of them be $90°$, the remaining side and angle will be indeterminate.

When $b = \mathrm{C}$, $\cot a = -\sin \mathrm{B}$; and if $b + \mathrm{C} = 180°$, $\cot a = \sin b$.

II. Given the hypothenusal angle and either of the sides, to find the angle adjacent to that side.

RULE.

As rad : tan hyp^{l}. ∠ : : cos given side : tan adjacent or required ∠.

Which ∠ is acute when the given side and hyp^{l}. ∠ are unlike; but obtuse (or the supplement of the former) when they are like.

Or,

$$\mathrm{Tan}\ \mathrm{A} = -\frac{\cos b \tan \mathrm{C}}{r};\ \tan \mathrm{B} = -\frac{\cos a \tan \mathrm{C}}{r}.$$

$\mathrm{L} \tan \mathrm{A} = \mathrm{L} \cos b + \mathrm{L} \tan \mathrm{C} - 10$; $\mathrm{L} \tan \mathrm{B} = \mathrm{L} \cos a + \mathrm{L} \tan \mathrm{C} - 10$.

When $b = \mathrm{C}$, $\tan \mathrm{A} = -\sin b$; and if $b + \mathrm{C} = 180°$, $\tan \mathrm{A} = \sin b$.

III. Given the hypothenusal angle and either of the sides, to find the angle opposite to that side.

RULE I.

As rad : sin hyp^{l}. ∠ : : sin given side : sin opposite or required ∠.

Which ∠ is acute when the opp. or given side is less than $90°$; but obtuse (or the supplement of the former) when it is greater than $90°$.

Or,

$$\mathrm{Sin}\ \mathrm{A} = \frac{\sin a \sin \mathrm{C}}{r};\ \sin \mathrm{B} = \frac{\sin b \sin \mathrm{C}}{r}.$$

$\mathrm{L} \sin \mathrm{A} = \mathrm{L} \sin a + \mathrm{L} \sin \mathrm{C} - 10$; $\mathrm{L} \sin \mathrm{B} = \mathrm{L} \sin b + \mathrm{L} \sin \mathrm{C} - 10$.

RULE II.

Find the other side by case I; then, by means of the hyp^l. ∠ and this side, find the ∠ adj^t. to it by case II; which will be the req^d. ∠, or that opp. the given side.

When $a = \mathrm{C}$, or $a + \mathrm{C} = 180°$, $\sin \mathrm{A} = \frac{1}{2}$ vers $2a$, or $\frac{\sin^2 a}{r}$.

IV. Given the hypothenusal angle and either of the other angles, to find the side adjacent to that angle.

RULE I.

As rad : cot hyp^l. ∠ : : tan given ∠ : cos adjacent or required side.

Which side is less than 90° when the hyp^l. ∠ and given ∠ are unlike; but greater than 90° (or the supplement of the former) when they are like.

Or,

$$\mathrm{Cos}\ a = -\frac{\tan \mathrm{B} \cot \mathrm{C}}{r};\ \cos b = -\frac{\tan \mathrm{A} \cot \mathrm{C}}{r}.$$

$\mathrm{L} \cos a = \mathrm{L} \tan \mathrm{B} + \mathrm{L} \cot \mathrm{C} - 10$; $\mathrm{L} \cos b = \mathrm{L} \tan \mathrm{A} + \mathrm{L} \cot \mathrm{C} - 10$.

RULE II.

$$\mathrm{Tan}\ \tfrac{1}{2} a = r \sqrt{\frac{\sin(\mathrm{C} + \mathrm{B})}{\sin(\mathrm{C} - \mathrm{B})}}$$

$$\mathrm{L} \tan \tfrac{1}{2} a = \frac{\complement\ \mathrm{L} \sin(\mathrm{C} \sim \mathrm{B}) + \mathrm{L} \sin(\mathrm{C} + \mathrm{B}) + 10}{2}.$$

The tan of $\frac{1}{2}b$ may also be expressed by the same form, using ∠ A instead of B; and, in either of these cases, the ½ side will be always less than 90°.

Note. In this, and the two following cases, if the given ∠ be less than the hypothenusal ∠, their sum will be less than 180°; and if it be greater than the hypothenusal ∠, their sum will be greater than 180°. If the two ∠ˢ be equal, the △ is indeterminate.

V. Given the hypothenusal angle and either of the other angles, to find the side opposite to that angle.

RULE I.

As sin hypˡ. ∠ : rad : : sin given ∠ : sin opposite or required side.

Which side is less than 90° when the opp. or given ∠ is acute; but greater than 90° (or the supplement of the former) when it is obtuse.

Or,

$$\mathrm{Sin}\ a = \frac{r \sin \mathrm{A}}{\sin \mathrm{C}}; \quad \sin\ b = \frac{r \sin \mathrm{B}}{\sin \mathrm{C}}.$$

$\mathrm{L} \sin a = \complement\,\mathrm{L} \sin \mathrm{C} + \mathrm{L} \sin \mathrm{A}$; $\mathrm{L} \sin b = \complement\,\mathrm{L} \sin \mathrm{C} + \mathrm{L} \sin \mathrm{B}$.

RULE II.

$$\mathrm{Tan}\,(45^\circ - \tfrac{1}{2}a) = \sqrt{\tan \tfrac{1}{2}(\mathrm{C} - \mathrm{A}) \cot \tfrac{1}{2}(\mathrm{C} + \mathrm{A})}$$

$$\mathrm{L} \tan\,(45^\circ \sim \tfrac{1}{2}a) = \frac{\mathrm{L} \tan \frac{1}{2}(\mathrm{C} \sim \mathrm{A}) + \mathrm{L} \cot \frac{1}{2}(\mathrm{C} + \mathrm{A})}{2}$$

The tan of $(45^\circ - \frac{1}{2}b)$ may also be expressed by the same form, using ∠ B instead of A; and the whole sides a or b, which are found by subtracting $2(45^\circ \sim \frac{1}{2}a)$, or $2(45^\circ \sim \frac{1}{2}b)$ from 90°, will be less or greater than 90°, according to the rule given above.

VI. Given the hypothenusal angle and either of the other angles, to find the remaining angle.

RULE I.

As cos given ∠ : rad : : cos hyp^{l}. ∠ : cos remaining or required ∠.

Which ∠ is acute if hyp^{l}. ∠ and given ∠ are unlike; but obtuse (or the supplement of the former) if they are like.

Or,

$$\text{Cos } A = -\frac{r \cos C}{\cos B}; \ \cos B = -\frac{r \cos C}{\cos A}.$$

$$L \cos A = \complement\, L \cos B + L \cos C; \ L \cos B = \complement\, L \cos A + L \cos C.$$

RULE II.

$$\text{Cot } \tfrac{1}{2} A = \sqrt{\tan \tfrac{1}{2} (C + B) \tan \tfrac{1}{2} (C - B)}$$

$$L \cot \tfrac{1}{2} A = \frac{L \tan \frac{1}{2} (C + B) + L \tan \frac{1}{2} (C \sim B)}{2}.$$

The cot of ½ B may also be expressed by the same form, using ∠ A instead of B; and in each of these cases the ½ ∠ will be always acute. If the two angles be equal, the triangle will be indeterminate.

VII. Given either of the sides and its adjacent angle, to find the hypothenusal angle.

RULE.

As cos given side : tan adj^{t}. or given ∠ : : rad : tan hypothenusal ∠.

Which hyp^{l}. ∠ is acute when the given side and ∠ are unlike; but obtuse (or the supplement of the former) when they are like.

Or,

$$\text{Tan } \mathrm{C} = -\frac{r \tan \mathrm{A}}{\cos b} = -\frac{r \tan \mathrm{B}}{\cos a}.$$

$\mathrm{L} \tan \mathrm{C} = € \, \mathrm{L} \cos b + \mathrm{L} \tan \mathrm{A} = € \, \mathrm{L} \cos a + \mathrm{L} \tan \mathrm{B}$.

Note. In this, and the two following cases, the given side and angle may be each of any magnitude under 180°.

When $b = \mathrm{A}$, $\tan \mathrm{C} = -\frac{r \tan b}{\cos b}$, and if $b + \mathrm{A} = 180°$, $\tan \mathrm{C} = \frac{r \tan b}{\cos b}$.

VIII. Given either of the sides and its adjacent angle, to find the other angle.

RULE.

As rad : sin given ∠ : : tan adj^t. or given side : tan remaining or required ∠.

Which ∠ is acute when opp. or given side is less than 90°; but obtuse (or the supplement of the former) when it is greater than 90°.

Or,

$$\text{Tan } \mathrm{A} = -\frac{\tan a \sin \mathrm{B}}{r}; \ \tan \mathrm{B} = -\frac{\tan b \sin \mathrm{A}}{r}.$$

$\mathrm{L} \tan \mathrm{A} = \mathrm{L} \tan a + \mathrm{L} \sin \mathrm{B} - 10$; $\mathrm{L} \tan \mathrm{B} = \mathrm{L} \tan b + \mathrm{L} \sin \mathrm{A} - 10$.

When $a = \mathrm{B}$, $\tan \mathrm{A} = -\frac{\tan a \sin a}{r}$; and if $a + \mathrm{B} = 180°$, $\tan \mathrm{A} = \frac{\tan a \sin a}{r}$.

IX. Given either of the sides and its adjacent angle, to find the other side.

RULE I.

As rad : sin given side : : cos adjacent or given ∠ : cos remaining or required side.

Which side is less than 90° when opp. or given ∠ is acute; but greater than 90° (or the supplement of the former) when it is obtuse.

Or,

$$\text{Cos } a = \frac{\sin b \cos A}{r}; \ \cos b = \frac{\sin a \cos B}{r}.$$

$L \cos a = L \sin b + L \cos A - 10$; $L \cos b = L \sin a + L \cos B - 10$.

RULE II.

Find the hyp^l. ∠ by case VII; then, by means of this angle and the given side, find the remaining or required side by case I.

When $b = A$, $\cos a = \frac{1}{2} \sin 2b$; and if $b + A = 180°$, $\cos a = -\frac{1}{2} \sin 2b$.

X. Given the two angles, to find either of the sides.

RULE.

As sin either ∠ : rad : : tan other ∠ : tan opposite or required side.

Which side is less than 90° when opp. or given ∠ is acute; but greater than 90° (or the supplement of the former) when it is obtuse.

Or,

$$\text{Tan } a = \frac{r \tan A}{\sin B}; \ \tan b = \frac{r \tan B}{\sin A}.$$

$L \tan a = \text{€} L \sin B + L \tan A$; $L \tan b = \text{€} L \sin A + L \tan B$.

Note. In this, and the following case, the two given angles may be each of any magnitude under 180°.

When $A = B$, $\tan A = \sec A$, or $\frac{r^2}{\cos A}$; and if $A + B = 180°$, $\tan a = -\sec A$, or $-\frac{r^2}{\cos A}$.

XI. Given the two angles, to find the hypothenusal angle.

RULE I.

As rad : cos either ∠ : : cos other ∠ : cos hyp^{l}. ∠.

Which hyp^{l}. ∠ is acute when the given angles are unlike; but obtuse (or the supplement of the former) when they are like.

Or,

$$\cos C = -\frac{\cos A \cos B}{r}.$$

$$L \cos C = L \cos A + L \cos B - 10.$$

RULE II.

Find either of the sides by case X; then, by means of this side and its adj^{t}. ∠, find the hyp^{l}. ∠ by case VII.

When $A = B$, $\cos C = -\frac{\cos^2 A}{r}$; and if $A + B = 180°$, $\cos C = \frac{\cos^2 A}{r}$.

XII. Given the two sides, to find the hypothenusal angle.

RULE I.

As rad : cot either given sides : : cot other side : cos hypothenusal ∠.

Which hypothenusal ∠ is acute when the given sides are unlike; but obtuse (or the supplement of the former) when they are like.

Or,

$$\text{Cos } \mathrm{C} = -\frac{\cot a \cot b}{r}.$$

$$\mathrm{L} \cos \mathrm{C} = \mathrm{L} \cot a + \mathrm{L} \cot b - 10.$$

RULE II.

$$\text{Tan } \tfrac{1}{2}\, \mathrm{C} = r \sqrt{\frac{\cos (a \sim b)}{-\cos (a + \mathrm{B})}}$$

$$\mathrm{L} \tan \tfrac{1}{2}\, \mathrm{C} = \frac{\text{€}\, \mathrm{L} \cos (a + b) + \mathrm{L} \cos (a \sim b) + 10}{2}.$$

Which ½ ∠ C is always less than 90°.

Note. The two sides, in this and the following case, must be so taken, that their difference shall be less than 90°, and their sum between 90° and 270°.

When $a = b$, $\cos \mathrm{C} = -\frac{\cot^2 a}{r}$; and if $\mathrm{A} + \mathrm{B} = 180°$, $\cos \mathrm{C} = \frac{\cot^2 a}{2}$.

XIII. Given the two sides, to find either of the angles.

RULE I.

As sin either given sides : rad : : cos other side : cos opposite or required ∠.

Which ∠ is acute when its opposite side is less than 90°; but obtuse (or the supplement of the former) when it is greater than 90°.

Or,

$$\text{Cos } \mathrm{A} = \frac{r \cos a}{\sin b}; \ \cos \mathrm{B} = \frac{r \cos b}{\sin a}.$$

L cos A = €L sin b + L cos a; L cos B = €L sin a + L cos b.

RULE II.

$$\text{Cot}\ \tfrac{1}{2}\,A = \sqrt{\cot\left(\frac{b+a}{2} - 45^\circ\right)\tan\left(\frac{b-a}{2} + 45^\circ\right)}$$

$$\text{L}\cot\tfrac{1}{2}\,A = \frac{\text{L}\cot\left(\frac{b+a}{2} - 45^\circ\right) + \text{L}\tan\left(\frac{b-a}{2} + 45^\circ\right)}{2}$$

The cot of ½ B may also be expressed by the same form, putting a in the place of b, and b in that of a; and the ½ angle, in each of these cases, is always less than 90°.

When $a = b$, cos A = cot a; and if $a + b = 180°$, cos A = − cot A.

XIV. Given either of the sides and its opposite angle, to find the other angle.

RULE I.

As rad : cot given side : : tan opp. or given ∠ : sin remaining or required ∠.

Which ∠ is ambiguous; that is, it may be either an acute angle, or its supplement.

Or,

$$\text{Sin}\ A = \frac{\cot b \tan B}{r};\ \sin B = \frac{\cot a \tan A}{r}.$$

L sin A = L cot b + L tan B − 10; L sin B = L cot a + L tan A − 10.

RULE II.

$$\text{Tan}\ (45^\circ - \tfrac{1}{2}A) = r\sqrt{\frac{\sin(b - B)}{\sin(b + B)}}$$

$$\text{L}\tan(45^\circ \sim \tfrac{1}{2}A) = \frac{€\text{L}\sin(b + B) + \text{L}\sin(b \sim B) + 10}{2}.$$

The tan of $(45° - \frac{1}{2}\mathrm{B})$ may also be expressed by the same form, using a, A instead of b, B; and A or B, which are found by subtracting $2\,(45° \sim \frac{1}{2}\mathrm{A})$ or $2\,(45° \sim \frac{1}{2}\mathrm{B})$ from 90°, are subject to the same ambiguity as in rule I.

Note. In this, and the two following cases, if the given angle be less than its opposite side, their sum will be less than 180°; and if it be greater than its opp. side, their sum will be greater than 180°. If the side and opp. angle be equal, the triangle is impossible.

XV. Given either of the sides and its opposite angle, to find the other side.

RULE.

As cos given ∠ : cos opp. or given side : : rad : sin remaining or required side.

Which side is ambiguous; that is, it may be either an arc less than 90°, or its supplement.

Or,

$$\mathrm{Sin}\ a = \frac{r\cos b}{\cos \mathrm{B}};\ \sin b = \frac{r\cos a}{\cos \mathrm{A}}.$$

$\mathrm{L}\sin a = \complement\,\mathrm{L}\cos \mathrm{B} + \mathrm{L}\cos b$; $\mathrm{L}\sin b = \complement\,\mathrm{L}\cos \mathrm{A} + \mathrm{L}\cos a$.

RULE II.

$$\mathrm{Tan}\ (45° - \tfrac{1}{2}a) = \sqrt{\tan\tfrac{1}{2}(b + \mathrm{B})\tan\tfrac{1}{2}(b - \mathrm{B})}$$

$$\mathrm{L}\tan(45° \sim \tfrac{1}{2}a) = \frac{\mathrm{L}\tan\frac{1}{2}(b + \mathrm{B}) + \mathrm{L}\tan\frac{1}{2}(b \sim \mathrm{B})}{2}$$

The tan of $(45° - \frac{1}{2}b)$ may also be expressed by the same form, using a, A instead of b, B; and a or b, which are found by subtracting 2 $(45^\circ - \frac{1}{2} a)$, or 2 $(45^\circ - \frac{1}{2} b)$ from 90°, is subject to the same ambiguity as in rule I.

XVI. Given either of the sides and its opposite angle, to find the hypothenusal angle.

RULE I.

As sin given side : sin opp. or given ∠ : : rad : sin hypothenusal ∠.

Which hyp^l. angle is ambiguous; that is, it may be either an acute angle or its supplement.

Or,

$$\text{Sin } C = \frac{r \sin A}{\sin a} = \frac{r \sin B}{\sin b}.$$

$$\text{L} \sin C = \text{€}\,\text{L} \sin a + \text{L} \sin A = \text{€}\,\text{L} \sin b - \text{L} \sin B.$$

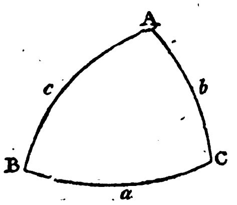
RULE II.

$$\text{Tan } (45^\circ - \tfrac{1}{2} C) = \sqrt{\cot \tfrac{1}{2} (a + A) \tan \tfrac{1}{2} (a - A)}.$$

$$\text{L} \tan (45^\circ \sim \tfrac{1}{2} C) = \frac{\text{L} \cot \frac{1}{2} (a + A) + \text{L} \tan \frac{1}{2} (a \sim A)}{2}.$$

The same form may also be applied to the other side and its opposite ∠, using b, B instead of a, A; and in each of these cases the hypothenusal ∠ C, which is found by subtracting 2 $(45^\circ \sim \frac{1}{2} C)$ from 90°, will be subject to the same ambiguity as in rule I.

SOLUTIONS OF ALL THE CASES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.



I. Given two sides and an angle opposite to one of them, to find the angle opposite the other.

RULE I.

As sin side opp. given ∠ : sin given ∠ : : sin other side : sin opposite or required ∠.

Which is either an acute ∠ or its supplement, according as it makes the greater ∠ opposite the greater side; and if each of them agree with this rule, the △ is ambiguous, or admits of two different solutions.

$$\text{Or, } \sin \text{A} = \frac{\sin a \sin \text{B}}{\sin b}.$$

$$\text{L} \sin \text{A} = \text{C L} \sin b + \text{L} \sin a + \text{L} \sin \text{B} - 10.$$

RULE II.

$$\text{Let C L} \sin b + \text{L} \sin a + \text{L} \sin \text{B} - 10 = \text{L} \tan \phi.$$

$$\text{Then, L} \tan (45^\circ \sim \tfrac{1}{2}\text{A}) = \frac{10 + \text{L} \tan (45^\circ - \phi)}{2};$$

Where arc ϕ can never be greater than 45°; and ∠ A, which is found by taking $2\,(45^\circ \sim \frac{1}{2}\text{A})$ from 90°, is subject to the same ambiguity as in rule I.

The ∠ B or C may also be expressed by the same form, using the sides and angle which are situated like those above; and the same observation may be extended to all the other cases of the table.

Note. In this, and the two following cases, the given sides and angle must be so taken, that the result, found by the 1st rule, shall not be greater than radius; otherwise the triangle is impossible.

When the two sides are equal, the req$^{\text{d}}$. ∠ will be equal to the given ∠; and if the sum of the two sides be 180°, the req$^{\text{d}}$. ∠ will be the supp$^{\text{t}}$. of the given ∠.

II. Given two sides and an angle opposite to one of them, to find the included angle.

RULE.

Find the angle opp. the other given side, by case I, and note whether it be ambiguous or not.

Then,

As sin $\frac{1}{2}$ diff. given sides : sin $\frac{1}{2}$ their sum :: tan $\frac{1}{2}$ diff. their opp. ∠s : cot $\frac{1}{2}$ included ∠.

Which $\frac{1}{2}$ ∠ is always acute; and if the ∠ found by the first part of the rule be ambiguous, the required ∠ will be ambiguous, otherwise not.

Or,

$$\text{Cot}\ \tfrac{1}{2}\ \text{C} = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a \sim b)} \tan \tfrac{1}{2}(\text{A} \sim \text{B}).$$

$\text{L} \cot \frac{1}{2}\,\text{C} = \text{C}\ \text{L} \sin \frac{1}{2}(a \sim b) + \text{L} \sin \frac{1}{2}(a+b) + \text{L} \tan \frac{1}{2}(\text{A} \sim \text{B}) - 10.$

Or $\frac{1}{2}$ the included ∠ may be found directly, by the following formula:

$$\text{Tan}\ \tfrac{1}{2}\ \text{C} = \frac{\sin a \cot \text{A}}{\sin (a+b)} \pm \sqrt{\frac{\sin^2 a \cot^2 \text{A}}{\sin^2 (a+b)} - \frac{r^2 \sin (a-b)}{\sin (a+b)}}.$$

When $a = b$, $\cot \frac{1}{2}\,\text{C} = \frac{\cos a \tan \text{A}}{r}$; and if $a + b = 180°$, $\tan \frac{1}{2}\,\text{C} = \frac{\cos a \tan \text{A}}{r}$.

III. Given two sides and an angle opposite to one of them, to find the remaining side.

RULE.

Find the angle opp. the other given side by case I, and note whether it be ambiguous or not.

Then,

As sin ½ diff. these $\angle^s$: sin ½ their sum : : tan ½ diff. given sides : tan ½ remaining side.

Which ½ side is always less than 90°; and if the ∠ found by the first part of the rule be ambiguous, the required side will be ambiguous, otherwise not.

Or,

$$\text{Tan}\ \tfrac{1}{2}\,c = \frac{\sin \frac{1}{2}\,(A + B)}{\sin \frac{1}{2}\,(A \sim B)} \tan \tfrac{1}{2}\,(a \sim b).$$

$\text{L} \tan \frac{1}{2}\,c = €\,\text{L} \sin \frac{1}{2}\,(A \sim B) + \text{L} \sin \frac{1}{2}\,(A + B) + \text{L} \tan \frac{1}{2}\,(a \sim b) - 10.$

Or ½ the remaining side may be found directly, by the following formula:

$$\text{Tan}\ \tfrac{1}{2}\,c = \frac{\sin b \cos A}{\cos a + \cos B} \pm \sqrt{\frac{\sin^2 b \cos^2 A}{(\cos a + \cos b)^2} + r^2\left(\frac{\cos a - \cos b}{\cos a + \cos b}\right)}.$$

When $a = b$, $\tan \frac{1}{2}\,c = \frac{\tan a \cos A}{r}$; and if $a + b = 180°$, $\cot \frac{1}{2}\,c = \frac{\tan a \cos A}{r}$.

IV. Given two angles and a side opposite to one of them, to find the side opposite the other.

RULE I.

As sin eith. given $\angle^s$: sin its opp. side : : sin other given ∠ : sin its opp. side.

Which side is an arc less than 90° or its supplement, according as it makes the greater side opp. the greater angle; and if each of them agree with this rule, the △ is ambiguous, or admits of two different solutions.

Or,

$$\text{Sin}\ a = \frac{\sin b \sin A}{\sin B}.$$

$\text{L} \sin a = €\,\text{L} \sin B + \text{L} \sin A + \text{L} \sin b - 10.$

RULE II.

Let $\text{€}\,\mathrm{L}\sin\mathrm{B} + \mathrm{L}\sin\mathrm{A} + \mathrm{L}\sin b - 10 = \mathrm{L}\tan\varphi$.

Then, $\mathrm{L}\tan(45^\circ \sim \tfrac{1}{2}a) = \dfrac{10 + \mathrm{L}\tan(45^\circ \sim \varphi)}{2}$.

Where arc φ is always less than 45°; and side a is subject to the same ambiguity as in rule I.

Note. In this, and the two following cases, the given $\angle^{s}$ and side must be so taken, that the result, found by the first rule, shall not be greater than radius, otherwise the triangle is impossible.

If the two $\angle^{s}$ be equal, the req^{d}. side will be equal to the given side; and if the sum of the two $\angle^{s}$ be 180°, the req^{d}. side will be the sup^{t}. of the given side.

V. Given two angles and a side opposite to one of them, to find the included side.

RULE.

Find the side opp. the other given angle by case IV, and note whether it be ambiguous or not.

Then,

As sin ½ diff. given $\angle^{s}$: sin ½ their sum : : tan ½ diff. their opp. sides : tan ½ included side.

Which ½ side is always less than 90°; and if the side found by the first part of the rule be ambiguous, the required side will be ambiguous, otherwise not.

Or,

$$\operatorname{Tan}\tfrac{1}{2}c = \frac{\sin\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\sin\frac{1}{2}(\mathrm{A}\sim\mathrm{B})}\tan\tfrac{1}{2}(a \sim b).$$

$\mathrm{L}\tan\tfrac{1}{2}c = \text{€}\,\mathrm{L}\sin\tfrac{1}{2}(\mathrm{A}\sim\mathrm{B}) + \mathrm{L}\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) + \mathrm{L}\tan\tfrac{1}{2}(a \sim b) - 10.$

Or ½ the included side may be found directly, by the following formula:

$$\text{Tan } \tfrac{1}{2} c = -\frac{\cot a \sin A}{\sin (a - B)} \pm \sqrt{\frac{\cot^2 a \sin^2 A}{\sin^2 (A - B)} + r^2 \times \frac{\sin (A + B)}{\sin (A - B)}}.$$

When $A = B$, $\tan \frac{1}{2} c = \frac{\tan a \cos A}{r}$; and if $A + B = 180°$, $\cot \frac{1}{2} c = \frac{\tan a \cos A}{r}$.

VI. Given two angles and a side opposite to one of them, to find the remaining angle.

RULE.

Find the side opp. the other given angle by case IV, and note whether it be ambiguous or not.

Then,

As sin ½ diff. these sides : sin ½ their sum : : tan ½ diff. given ∠s : cot ½ remaining ∠.

Which ½ ∠ is always acute; and if the side found by the first part of the rule be ambiguous, the required angle will be ambiguous, otherwise not.

Or,

$$\text{Cot } \tfrac{1}{2} C = \frac{\sin \frac{1}{2} (a + b)}{\sin \frac{1}{2} (a \sim b)} \tan \tfrac{1}{2} (A \sim B).$$

$L \cot \frac{1}{2} C = C\,L \sin \frac{1}{2} (a \sim b) + L \sin \frac{1}{2} (a + b) + L \tan \frac{1}{2} (A \sim B) - 10.$

Or ½ the remaining angle may be found directly, by the following formula:

$$\text{Tan } \tfrac{1}{2} C = -\frac{\cos b \sin A}{\cos A - \cos B} \pm \sqrt{\frac{\cos^2 b \sin^2 A}{(\cos A - \cos B)^2} + r^2 \times \frac{\cos A + \cos B}{\cos A - \cos B}}$$

When $\mathrm{A} = \mathrm{B}$, $\cot \frac{1}{2}\,\mathrm{C} = \frac{\cos a \tan \mathrm{A}}{r}$; and if $\mathrm{A} + \mathrm{B} = 180^\circ$, $\tan \frac{1}{2}\,\mathrm{C} = \frac{\cos a \tan \mathrm{A}}{r}$.

VII. Given two sides and their included angle, to find either of the other angles.

RULE.

As cos ½ sum given sides : cos ½ their diff. : : cot ½ their included ∠ : tan ½ sum other two $\angle^s$.

And,

As sin ½ sum given sides : sin ½ their diff. : : cot ½ their included ∠ : tan ½ diff. other two $\angle^s$.

Where ½ the sum of the two $\angle^s$ is like ½ the sum of their opposite sides; and ½ their difference is always less than 90°.

Then, if ½ the difference of these two $\angle^s$ be added to ½ their sum, it will give the ∠ opposite the greater side; and if subtracted from the ½ sum, it will give the ∠ opposite the less side.

Or,

$$\operatorname{Tan} \tfrac{1}{2}\,(\mathrm{A} + \mathrm{B}) = \frac{\cos \frac{1}{2}\,(a \sim b)}{\cos \frac{1}{2}\,(a + b)} \cot \tfrac{1}{2}\,\mathrm{C}.$$

$$\operatorname{Tan} \tfrac{1}{2}\,(\mathrm{A} \sim \mathrm{B}) = \frac{\sin \frac{1}{2}\,(a \sim b)}{\sin \frac{1}{2}\,(a + b)} \cot \tfrac{1}{2}\,\mathrm{C}.$$

$\mathrm{L} \tan \frac{1}{2}\,(\mathrm{A}+\mathrm{B}) = \text{Є}\,\mathrm{L} \cos \frac{1}{2}\,(a+b) + \mathrm{L} \cos \frac{1}{2}\,(a \sim b) + \mathrm{L} \cot \frac{1}{2}\,\mathrm{C} - 10.$

$\mathrm{L} \tan \frac{1}{2}\,(\mathrm{A} \sim \mathrm{B}) = \text{Є}\,\mathrm{L} \sin \frac{1}{2}\,(a+b) + \mathrm{L} \sin \frac{1}{2}\,(a \sim b) + \mathrm{L} \cot \frac{1}{2}\,\mathrm{C} - 10.$

Or either angle may be found directly, by the following formula:

$$\operatorname{Cot} \mathrm{A} = \frac{\cot a \sin b - \cos b \cos \mathrm{C}}{\sin \mathrm{C}}.$$

Note. In this, and the two following cases, the two given sides and their included angle may be each of any magnitude under 180°.

When $a = b$, $\cot \mathrm{A} = \frac{\cos a \tan \frac{1}{2} \mathrm{C}}{r}$; and if $a + b = 180°$, $\cot \mathrm{A} = \frac{\cos a \cot \frac{1}{2} \mathrm{C}}{r}$.

VIII. Given two sides and their included angle, to find the remaining side.

RULE I.

As rad : cos given ∠ : : tan either given sides : tan arc φ.

Then,

As cos arc φ : cos side above used : : cos diff. other side and φ : cos remaining or required side.

Where arc φ is less than 90°, when the given angle and side, used in the first part of the rule, are like; but greater than 90° when they are unlike.

The required side is also less than 90°, when the given angle and the diff. of the other side and arc φ are like; but greater than 90° when they are unlike.

Or, put $\mathrm{L} \tan b + \mathrm{L} \cos \mathrm{C} - 10 = \mathrm{L} \tan \varphi$.

Then, $\mathrm{L} \cos c = \text{€}\,\mathrm{L} \cos \varphi + \mathrm{L} \cos b + \mathrm{L} \cos (a \sim \varphi) - 10$.

RULE II.

Find the two remaining angles by case VII.

Then, As sin ½ diff. these ∠s : sin ½ their sum : : tan ½ diff. their opp. sides : tan ½ remaining side.

Which ½ side is always less than 90°.

Or,

$$\operatorname{Tan} \tfrac{1}{2} c = \frac{\sin \frac{1}{2} (\mathrm{A} + \mathrm{B})}{\sin \frac{1}{2} (\mathrm{A} \sim \mathrm{B})} \tan \tfrac{1}{2} (a \sim b).$$

$\text{L} \tan \frac{1}{2} c = \text{C L} \sin \frac{1}{2} (\text{A} \sim \text{B}) + \text{L} \sin \frac{1}{2} (\text{A} + \text{B}) + \text{L} \tan \frac{1}{2} (a \sim b) - 10.$

Or the remaining side may be found by the following formula:

$$\text{Cos } c = \frac{r \cos a \cos b + \sin a \sin b \cos \text{C}}{r}.$$

When $a = b$, $\sin \frac{1}{2} c = \frac{\sin a \sin \frac{1}{2} \text{C}}{r}$; and if $a + b = 180°$, $\cos \frac{1}{2} c = \frac{\sin a \cos \frac{1}{2} \text{C}}{r}$.

IX. Given two angles and their included side, to find either of the opposite sides.

RULE.

As cos ½ sum given $\angle^s$: cos ½ their diff. : : tan ½ included side : tan ½ sum other two sides.

And,

As sin ½ sum given $\angle^s$: sin ½ their diff. : : tan ½ included side : tan ½ diff. other two sides.

Where ½ the sum of the two sides is like ½ the sum of their opp. $\angle^s$; and ½ their diff. is less than 90°.

And if ½ the diff. of the sides be added to ½ their sum, it will give the side opposite the greater $\angle$, and if it be subtracted from the ½ sum it will give the side opposite the less $\angle$.

Or,

$$\text{Tan } \frac{1}{2} (a + b) = \frac{\cos \frac{1}{2} (\text{A} \sim \text{B})}{\cos \frac{1}{2} (\text{A} + \text{B})} \tan \frac{1}{2} c.$$

$$\text{Tan } \frac{1}{2} (a \sim b) = \frac{\sin \frac{1}{2} (\text{A} \sim \text{B})}{\sin \frac{1}{2} (\text{A} + \text{B})} \tan \frac{1}{2} c.$$

$\mathrm{L}\tan\frac{1}{2}(a+b) = \mathrm{C}\,\mathrm{L}\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}) + \mathrm{L}\cos\frac{1}{2}(\mathrm{A}\sim\mathrm{B}) + \mathrm{L}\tan\frac{1}{2}c - 10.$

$\mathrm{L}\tan\frac{1}{2}(a\sim b) = \mathrm{C}\,\mathrm{L}\sin\frac{1}{2}(\mathrm{A}+\mathrm{B}) + \mathrm{L}\sin\frac{1}{2}(\mathrm{A}\sim\mathrm{B}) + \mathrm{L}\tan\frac{1}{2}c - 10.$

Or the side may be found directly, by the following formula:

$$\mathrm{Cot}\ a = \frac{\cot \mathrm{A} \sin \mathrm{B} + \cos c \cos \mathrm{B}}{\sin c}.$$

Note. In this, and the following case, the two given ∠ˢ and their included side may be each of any magnitude under 180°.

When $\mathrm{A} = \mathrm{B}$, $\cot a = \frac{\cot \frac{1}{2} c \cos \mathrm{A}}{r}$; and if $\mathrm{A} + \mathrm{B} = 180°$, $\cot a = \frac{\tan \frac{1}{2} c \cos \mathrm{A}}{r}$.

X. Given two angles and their included side, to find the remaining angle.

RULE I.

As rad : cos given side : : tan either given ∠ˢ : cot arc φ.

Then,

As sin arc φ : cos ∠ above used : : sin diff. other ∠ and φ : cos remaining or required ∠.

Where arc φ is less than 90° when the given side and angle, used in the first part of the rule, are like; but greater than 90° when they are unlike.

The required angle is also like the angle above mentioned, when arc φ is less than the other given angle; but unlike it when it is greater.

Or,

$$\frac{\tan \mathrm{A} \cos c}{r} = \cot \varphi; \text{ then } \cos \mathrm{C} = \frac{\cos \mathrm{A} \sin (\mathrm{B} \sim \varphi)}{\sin \varphi}.$$

Or,

$$\mathrm{L} \tan \mathrm{A} + \mathrm{L} \cos c - 10 = \mathrm{L} \cot \varphi.$$

Then, $\mathrm{L} \cos \mathrm{C} = \complement \mathrm{L} \sin \varphi + \mathrm{L} \sin (\mathrm{B} \sim \varphi) + \mathrm{L} \cos \mathrm{A} - 10$.

RULE II.

Find the two remaining sides by case IX.

Then,

As sin $\frac{1}{2}$ diff. these sides : sin $\frac{1}{2}$ their sum : : tan $\frac{1}{2}$ diff. given ∠ˢ : cot $\frac{1}{2}$ remaining ∠.

Which $\frac{1}{2}$ ∠ is always less than 90°.

Or,

$$\operatorname{Cot} \tfrac{1}{2} c = \frac{\sin \frac{1}{2} (a + b)}{\sin \frac{1}{2} (a \sim b)} \tan \tfrac{1}{2} (\mathrm{A} \sim \mathrm{B}).$$

$\mathrm{L} \cot \frac{1}{2} \mathrm{C} = \complement \mathrm{L} \sin \frac{1}{2} (a \sim b) + \mathrm{L} \sin \frac{1}{2} (a + b) + \mathrm{L} \tan \frac{1}{2} (\mathrm{A} \sim \mathrm{B}) - 10$.

Or the remaining angle may be found directly, by the following formula:

$$\operatorname{Cos} \mathrm{C} = \frac{\cos c \sin \mathrm{A} \sin \mathrm{B} - r \cos \mathrm{A} \cos \mathrm{B}}{r^2}.$$

When $\mathrm{A} = \mathrm{B}$, $\cos \frac{1}{2} \mathrm{C} = \frac{\cos \frac{1}{2} c \sin \mathrm{A}}{r}$; and if $\mathrm{A} + \mathrm{B} = 90°$, $\sin \frac{1}{2} \mathrm{C} = \frac{\cos \frac{1}{2} c \sin \mathrm{A}}{r}$.

XI. Given the three sides to find either of the angles.

RULE.

As rect. under the sine of $\frac{1}{2}$ the sum of the three sides and the sine of the difference between this $\frac{1}{2}$ sum and the side opp. the ∠ sought, is to the square of rad.

So is rect. under the sines of the differences of the same ½ sum and each of the other two sides, to the square of the tan of ½ required ∠.

Or,

If *s* be made to denote the sum of the three sides, *c* the side opp. the required ∠, and *a*, *b* the sides about that ∠.

Then,

$$\text{Tan}\ \tfrac{1}{2}\,\text{c} = r\sqrt{\frac{\sin\left(\frac{1}{2}s - a\right)\sin\left(\frac{1}{2}s - b\right)}{\sin\frac{1}{2}s\,\sin\left(\frac{1}{2}s - c\right)}}.$$

$$\text{L}\tan\tfrac{1}{2}\text{c} = \frac{\mathrm{CL}\sin\frac{1}{2}s + \mathrm{CL}\sin\left(\frac{1}{2}s - c\right) + \mathrm{L}\sin\left(\frac{1}{2}s - a\right) + \mathrm{L}\sin\left(\frac{1}{2}s - b\right)}{2}$$

Which ½ ∠ is always acute.

Or this logarithmic formula may be expressed in words, as follows:

Add together the log sine of ½ the sum of the three sides, and the log sine of the difference between this ½ sum and the side opp. the angle sought, and find the complement of their sum, by taking the index from 19, and the rest of the figures from 9, as usual.

Then to this complement add the log sines of the differences between the same ½ sum and each of the other two sides, and the result, divided by 2, will give the log tan of ½ the required ∠.

Or the whole angle may be found by the following formula:

$$\text{Cos c} = \frac{r^2 \cos c - r \cos a \cos b}{\sin a \sin b}.$$

Note. In this case the three sides must be so taken, that the sum of any two of them may be greater than the third, and the sum of all three of them less than 360°, When all the three sides of the △ are equal, the cos of either ∠, as $\mathrm{C} = \frac{\cot a \tan \frac{1}{2} a}{r}$; or $\sin \frac{1}{2} \mathrm{C} = \frac{r \sin \frac{1}{2} a}{\sin a} = \frac{1}{2} \sec \frac{1}{2} \mathrm{A}$.

If two of the three sides only are equal, $\sin \frac{1}{2} \mathrm{C} = \frac{r \sin \frac{1}{2} c}{\sin a}$; and when the sum of two of the sides is 180°, $\cos \frac{1}{2} \mathrm{C} = \frac{r \cos \frac{1}{2} c}{\sin a}$.

XII. Given the three angles, to find either of the sides.

RULE.

As rect. under the cosine of ½ the sum of the three $\angle^s$ and the cosine of the diff. between this ½ sum and the ∠ opp. the side sought, is to the square of rad.

So is the rect. under the cosines of the differences of the same ½ sum and each of the other two $\angle^s$, to the square of the cot ½ required side.

Or,

If S be made to denote the sum of the three $\angle^s$, C the ∠ opp. the required side, and A, B the $\angle^s$ adj^t. to that side.

Then,

$$\operatorname{Cot} \tfrac{1}{2} c = r\sqrt{\frac{\cos\left(\frac{1}{2}\mathrm{S} - \mathrm{A}\right) \cos\left(\frac{1}{2}\mathrm{S} - \mathrm{B}\right)}{-\cos \frac{1}{2}\mathrm{S} \cos\left(\frac{1}{2}\mathrm{S} - \mathrm{C}\right)}}.$$

$$\mathrm{L} \cot \tfrac{1}{2} c = \frac{\complement \mathrm{L} \cos \frac{1}{2}\mathrm{S} + \complement \mathrm{L} \cos\left(\frac{1}{2}\mathrm{S} - \mathrm{C}\right) + \mathrm{L} \cos\left(\frac{1}{2}\mathrm{S} - \mathrm{A}\right) + \mathrm{L} \cos\left(\frac{1}{2}\mathrm{S} - \mathrm{B}\right)}{2}$$

Which ½ side is always less than 90°.

Or this logarithmic formula may be expressed in words, as follows:

Add together the log cosine of ½ the sum of the three $\angle^s$ and the log cosine of the diff. between this ½ sum and the $\angle$ opp. the side sought, and find the complement of their sum, by taking the index from 19, and the rest of the figures from 9, as usual.

Then to this complement add the log cosines of the differences between the same ½ sum and each of the other two $\angle^s$, and the result, divided by 2, will give the log cot. of ½ the required side.

Or the whole side may be found by the following formula:

$$\text{Cos } c = \frac{r^2 \cos \text{C} + r \cos \text{A} \cos \text{B}}{\sin \text{A} \sin \text{B}}.$$

Note. In this case the three angles must be so taken, that the sum of any two of them may be greater than the supplement of the third, and that the sum of all three of them may fall between 180° and 540°.

When all the three $\angle^s$ of the $\triangle$ are equal, the cos of either side, as $c \cos = \frac{\cot \text{C} \cot \frac{1}{2} \text{C}}{r}$, or $\cos \frac{1}{2} \text{c} = \frac{r \cos \frac{1}{2} \text{C}}{\sin \frac{1}{2} \text{C}} = \frac{1}{2} \operatorname{cosec} \frac{1}{2} \text{C}$.

If two of the three $\angle^s$ only are equal, $\cos \frac{1}{2} \text{c} = \frac{r \cos \frac{1}{2} \text{C}}{\sin \text{A}}$; and when the sum of two of the $\angle^s$ is 180°, $\sin \frac{1}{2} c = \frac{r \sin \frac{1}{2} \text{C}}{\sin \text{A}}$ (*q*).

(*q*) As every isosceles spherical $\triangle$ is divided into two equal right-angled spherical $\triangle^s$, by a perpendicular drawn from the vertical $\angle$ to the base, it is evident that the solutions of all the cases of the former, as given in the foregoing Tables, may be derived from those of the latter, by referring them to the right-angled $\triangle$ to which they belong.

Also, in the species of $\triangle$ which has the sum of two of its sides

MISCELLANEOUS QUESTIONS FOR EXERCISING THE RULES IN THE PRECEDING TABLES.

1. In a right-angled spherical triangle A B C, one of the oblique angles A is 60°, and the other B 45°; required the side B C opposite the former. Ans. B C 45°.

2. In a right-angled isosceles spherical triangle A B C, the two equal sides A C, C B are each 30°; required the hypothenuse A B. Ans. A B 41° 24′ 35″.

3. It is required to find the angles of an equilateral spherical triangle A B C, each side of which is 60°. Ans. each ∠ 70° 31′ 44″.

4. In a quadrantal spherical triangle A B C, the hypothenusal angle C is 87½°, and one of the other angles A 95° 13′; required the side A C adjacent to that angle. Ans. A C 61° 25′ 53″.

5. In a right-angled spherical triangle A B C, the hypothenuse A B is 84° 23′ 20″, and one of the legs A C 96° 36′ 22″; required the angle A adjacent to that leg. Ans. ∠ A 148° 1′ 49″.

6. In an isosceles spherical triangle A B C, each of the two equal sides A B, A C are 95½°, and their included angle 100°; required the base B C. Ans. B C 99° 22′ 24″.

7. In an oblique-angled spherical triangle A B C, one of its sides A B is 96° 50′, the other A C 83° 10′,

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equal to 180°, the sum of their opposite ∠ˢ will likewise be equal to 180°; and, consequently, if the base and one of these sides be produced to semicircles, or till they meet, the supplemental △, thus formed, will be isosceles, and can, therefore, be resolved by some of the cases of right-angled triangles, in a similar manner with the former.

and their included angle A 120°; required the base B C, and the other two angles B and C.

Ans. B C 120° 28′ 10″, ∠ B 86° 4′ 13″,
∠ C 93° 55′ 47″.

8. In a right-angled spherical triangle A B C, the hypothenuse A B is 61° 4′ 56″, and the leg B C 40° 30′ 20″; required the other leg A C, and the angles A and B.

Ans. A C 50° 30′ 29″, ∠ A 47° 54′ 21″, ∠ B 61° 50′ 28″.

9. In the quadrantal spherical triangle A B C, the side B C is 132° 5′ 40″, and the side A C 118° 9′ 31″; it is required to find the hypothenusal angle C, and the other two angles A and B.

Ans. C = 118° 55′ 4″, ∠ A 139° 29′ 41″,
∠ B 129° 29′ 30″.

10. In the oblique-angled spherical triangle A B C, the side A B is 76° 35′ 36″, A C 50° 10′ 30″, and B C 40° 0′ 10″; required the three angles A, B and C.

Ans. ∠ A 34° 15′ 3″, ∠ B 42° 15′ 13¼″,
∠ C 121° 36′ 20″.

11. At each of three objects, on the surface of the earth, the angles subtended by the other two were 54° 29′ 36″, 93° 32′ 59″, and 31° 57′ 25″ respectively, from which it is required to find the distances of the objects from each other.

Ans. 106213, 130224.4, and 69058 feet.

Note. A great variety of examples of this kind may be found in that part of the Trigonometrical Survey of England and Wales already published; in which, as well as in several other similar performances, these kind of geodetical operations are amply detailed.

APPLICATION OF SPHERICAL TRIGONOMETRY TO THE RESOLUTION OF ASTRONOMICAL PROBLEMS.

Astronomical Problems are such as chiefly relate to the determination of the relative situations and positions of the heavenly bodies with respect to each other, or to certain imaginary points, lines, and circles of the sphere, that have been fixed upon, by astronomers and geographers, for this purpose; each of which belong equally to the globe of the earth and the concave sphere of the heavens that surrounds it, and are distinguished as follows:

The axis of the celestial sphere, is an imaginary line passing through the centre of the earth, about which the sun and stars appear to perform their daily revolutions (*r*).

The north and south poles of the celestial sphere, are the two extremities of its axis; and the points lying directly under them, on the terrestrial sphere, are *the north and south poles of the earth* (*s*).

(*r*) Although it is the diurnal rotation of the earth upon its axis in 24 hours, and its annual revolution round the sun in 365 days, 5 hours, 48 minutes, 51.6 seconds, which occasions the various phænomena of the heavens, yet as the places and appearances of the celestial bodies will be the same, whether the earth moves round the sun, or the sun and the rest of the celestial bodies move round the earth, astronomers, for the ease of calculation, usually assume the earth as a point at rest in the centre of the system, and ascribe to the heavenly bodies that motion which they appear to have to a spectator on its surface.

(*s*) The star commonly called the north-pole star is not directly

The equinoctial, or celestial equator, is a great circle which divides the northern half of the heavens from the southern; and, when referred to the earth, it is called simply *the equator*.

The ecliptic is a great circle which cuts the equator in two opposite points, at an angle of 23° 28′ nearly, being that in which the sun appears to perform his annual revolution through the heavens (*t*).

This circle, like all others, is divided into 360°, and also into twelve equal parts, of 30 degrees each, called signs, the names and characters of which are as follows:

N.	Aries	Taurus	Gemini	Cancer	Leo	Virgo
	♈	♉	♊	♋	♌	♍
S.	Libra	Scorpio	Sagittarius	Capricornus	Aquarius	Pisces
	♎	♏	♐	♑	♒	♓

The sun continues about a month in each of these signs, and goes through nearly a degree in a day.

Meridians are great circles passing through the poles of the world, and cutting the celestial equator at right angles.

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in the point which is the true north pole of the heavens; its declination, for the beginning of the year 1790, having been found to be 88° 11′ 8″, or 1° 48′ 52″ from the pole, and its annual variation 19½″.

(*t*) It has been found, by comparing the observations of the ancients with those of the moderns, as well as from physical principles, that the obliquity of the ecliptic, or the angle which it makes with the equator, is continually diminishing towards a certain limit; and that it will, afterwards, return again; but the variation is very small, and has been differently stated by different authors.

They are also called *hour circles*, or *circles of right ascension;* and, when referred to the earth, *circles of terrestrial longitude.*

The horizon is a circle which is distinguished by several names, according to the sense in which it is employed: thus,

The rational, or *true horizon*, is a great circle which divides the upper half of the heavens from the lower; being formed by a plane passing through the centre of the earth, at the distance of 90°, each way, from the spectator.

The astronomical horizon, is an imaginary plane, touching the earth in the point supposed to be the centre of view of the spectator, and extending indefinitely on all sides; being that to which the risings and settings of the heavenly bodies are always referred.

Where it is to be observed, that this latter horizon, with regard to the fixed stars, coincides, as to sense, with the rational horizon; but for the sun, their distance from each other is about 9 seconds, and for the moon about 33 minutes.

The sensible or *visible horizon* is a small circle of the earth, parallel to the rational horizon, and is the boundary of the spectator's view at sea or land; being more or less extended, according to the height from which it is observed (*u*).

It may also be remarked, that the first of the three

(*u*) It may here be further observed, that the astronomical, as well as the sensible horizon, is frequently referred to a parallel plane passing through the earth's centre; in which case, it is then called the reduced horizon.

circles here mentioned is divided, by mariners, into 32 equal parts, of 11° 15′ each, called *the points of the compass;* the principal of which, *east, west, north* and *south,* are denominated *the four cardinal points.*

The zodiac is a space which extends about 8° on each side of the ecliptic, like a belt or girdle, being that in which the paths or orbits of the principal planets are situated.

Circles of celestial longitude are great circles passing through the poles of the ecliptic, and cutting it at right angles.

The tropics are two small circles, at the distance of 23° 28′ from the equator; that in the northern hemisphere being called *the tropic of cancer,* and the other, in the southern hemisphere, *the tropic of capricorn.*

The polar circles are two small circles at the distance of 23° 28′ from the poles; that in the northern hemisphere being called *the arctic circle,* and the other, in the southern hemisphere, *the antarctic circle.*

All circles of this kind, which are parallel to the equator, are also called parallels of terrestrial latitude, or of declination.

Parallels of celestial latitude are small circles parallel to the ecliptic; and *parallels of altitude,* or *almacanters,* are small circles parallel to the horizon.

The equinoctial points are the two points where the ecliptic cuts the equinoctial; and the *solsticial points* are the two points where it touches the tropics (*x*).

(*x*) When the sun appears in either of the equinoctial points the days and nights are equal, or 12 hours each, all over the world;

The zenith and nadir are the two poles of the horizon; the former being the point directly over our heads, and the latter that under our feet.

The equinoctial colure is a meridian passing through the equinoctial points; and *the solsticial colure* is a meridian passing through the solsticial points.

Azimuth, or *vertical circles*, are great circles passing through the zenith and nadir of any place, and cutting the horizon at right angles.

The prime vertical, is the azimuth circle which passes through the east and west points of the horizon; being that on which the sun rises and sets when the days and nights are equal.

The six o'clock hour-circle, is that meridian which cuts the 12 o'clock hour-circle, or meridian of the place, at right angles; being that on which the sun is at six o'clock, both in the morning and evening.

The latitude of any celestial object, is its distance north or south from the ecliptic, as measured on a circle of celestial longitude passing through its centre.

The latitude of any place on the earth, is its distance

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which happens about the 21st of March and the 22d of September; the former being called *the vernal equinox*, and the latter *the autumnal equinox*.

Also, when the sun comes to the northern solsticial point, or tropic of cancer, it is the longest day; and when he comes to the southern solsticial point, or tropic of capricorn, it is the shortest day; the former of these happening about the 21st of June, and the latter about the 22d of December.

The solsticial points cancer (♋) and capricorn (♑) are so called, because when the sun is in this situation he seems to have nearly the same altitude at noon for several days together, and on that account appears to stop or stand still.

north or south from the equator, as measured on a meridian passing through that place.

The longitude of any celestial object, is the distance of the circle of longitude, which passes through its centre, from the first point of aries, as reckoned in degrees, minutes, &c. quite round the ecliptic (*y*).

The longitude of any place on the earth, is the distance east or west of the meridian, which passes over that place, from the first meridian, reckoned in degrees, minutes, &c. on the equator.

Note. The first meridian, in this country, is that which passes over the Royal Observatory at Greenwich; and for other countries, it is that of the capital of the kingdom.

The co-latitude, or *polar distance*, of any place on the earth, is an arc of a meridian, contained between that place and the north or south pole of the world.

The co-latitude, or *polar distance*, of any celestial object is an arc of a circle of celestial longitude contained between the centre of that object and the north or south pole of the ecliptic.

The dip of the horizon is the angle formed by an horizontal line, passing through the eye of the ob-

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(*y*) The sun being always situated in some part of the ecliptic, has no latitude: and instead of his longitude, his distance in signs, degrees, minutes, &c. from the first point of aries, is sometimes called his place in the ecliptic.

It may also be observed, that the latitude of any place is equal to the height of the pole above the horizon, or to the distance of the zenith of that place from the celestial equator, as counted upon the meridian of that place.

server, and another line drawn so as to be a tangent to the earth, at some point on its surface (z).

Refraction is the difference between the real and apparent place of an object, as occasioned by the rays of light passing through the atmosphere in a curvilinear instead of a right-lined direction.

Parallax is the difference between the places of any celestial object as seen from the surface of the earth and from its centre.

Horizontal parallax is the angle under which the semidiameter of the earth would appear if seen directly from the centre of the sun, or a planet.

Parallax in altitude is the angle under which the semidiameter of the earth would appear if seen obliquely from the centre of the sun, or a planet (*a*).

The altitude of any of the heavenly bodies, is an arc of an azimuth, or vertical, circle, contained between the centre of the object and the horizon.

(z) As the ∠ of depression, or dip of the horizon, occasions objects to appear higher than they really are, it must be subtracted from the observed altitude, in order to obtain the apparent altitude.

(*a*) The more elevated a planet is above the horizon, the less is its parallax, its distance from the earth's centre continuing the same. When the planet is in the zenith it has no parallax, and when in the horizon its parallax is the greatest.

Refraction and the dip of the horizon make all objects appear higher than they really are; but parallax, having a contrary effect, makes them appear lower. The fixed stars are at such immense distances that they have no sensible parallax.

The observed altitude is that which is taken simply with a sextant, or other instrument, without being corrected for the dip of the horizon, refraction, or parallax.

The apparent altitude is that which has been corrected for the dip of the horizon, without considering the effect of refraction or parallax.

The true altitude is that which is found after making all the corrections arising from the dip of the horizon, refraction, and parallax.

The zenith distance is the complement of the altitude, or the arc of a vertical circle, contained between the centre of the object and the zenith.

The declination of the sun, moon, or stars, is an arc of a meridian contained between the centre of that object and the celestial equator.

The amplitude of any of the heavenly bodies, is an arc of the horizon contained between the centre of the object, when rising or setting, and the east or west points of the horizon.

The azimuth of any of the heavenly bodies, is an arc of the horizon contained between an azimuth circle, passing through the centre of the object, and the north or south points of the horizon (*b*).

The right ascension of any celestial body, is an arc of the equinoctial contained between the first point of

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(*b*) *The amplitude* of any of the heavenly bodies may also be considered as its bearing, when in the horizon, from the true east or west point of the compass; and *the azimuth*, as its bearing from one of these points, when it is at any height above the horizon.

aries and a meridian passing through the centre of the object (c).

Or, it is that measure of time which shows how much sooner, or later, than the first point of aries any star comes to the meridian; and consequently the difference of time between one star coming to the meridian and another.

The oblique ascension, or *descension*, is the distance of the first point of aries from the horizon when the object is rising or setting, as reckoned on the equator; or it is that point of the celestial equator which rises or sets with the object in an oblique sphere.

The ascensional difference is the difference between the right and oblique ascension or descension; or it is the time that the sun rises before six o'clock in the summer, or sets before six in the winter.

Aberration is an apparent change of place in the stars, arising from the motion of the earth combined with the motion of light.

The achronical rising or setting of a planet, or star, is when it rises at sun-set, or sets at sun-rise; and the *cosmical rising or setting* is when it rises or sets with the sun.

The heliacal rising of a star, is when it appears above the horizon just before the sun in the morning; and its *heliacal setting*, is when it sinks below the horizon immediately after him in the evening.

(c) The sun's longitude and right ascension, are usually reckoned from the first point of aries, quite round the globe; hence, although at equal distances from the equinoctial points, his declination may be of the same name, N. or S., and the same quantity, yet the longitudes and right ascensions will be materially different.

Time is a portion of duration, which, in astronomical computations, is divided into three kinds, viz. apparent, true or mean time, and sidereal time.

Apparent time is that shown by the sun; being reckoned from the instant of his passing over any meridian, in one day, to the instant of his passing over it the next.

Sidereal time is that which is measured by a well-regulated clock or watch, so adjusted as to count 24 hours from the passage of a star over any meridian till it returns to that meridian again.

True, or *mean time*, is that which is measured by a clock or watch, so adjusted as to count $24^h\ 3^m\ 56\frac{1}{2}^s$ in a mean solar day, or $365^d\ 5^h\ 48^m\ 51.6^s$ in a mean solar year.

True, or *mean noon*, is 12 o'clock, as shown by a well-regulated chronometer, so adjusted as to go 24 hours in a mean solar day; and *apparent noon*, is when the sun comes to the meridian of the place, as shown by the same machine (*d*).

(*d*) It may also be remarked, that besides the divisions of time above mentioned, there are four kinds of lunar months, and three kinds of solar years, which are distinguished as follows:

The periodical month is the period in which the moon returns to the first point of aries, consisting of $27^d\ 7^h\ 43^m\ 5^s$.

The sidereal month is the period in which the moon returns to the same star, consisting of $27^d\ 7^h\ 43^m\ 12^s$.

The synodical month is the period in which the moon returns to the sun, consisting of $29^d\ 12^h\ 44^m\ 3^s$.

The right ascension of the mid heaven is the distance of the first point of aries from the meridian, at the time and place of observation, as reckoned on the equator.

The nonagesimal degree, or *medium cœli*, is the 90th degree of the ecliptic, reckoned from its intersection with the eastern part of the horizon at any given time.

Its altitude is equal to the distance between the zenith and the pole of the ecliptic; or it is measured by the angle which the ecliptic makes with the horizon, at any elevation of the pole.

The crepusculum circle, is a small circle parallel to the horizon, at the distance of about 18° below it, where the twilight is supposed to begin and end.

The rising of any celestial object, is when its centre appears in the eastern part of the horizon; and its *setting* is when its centre comes to the western part of the horizon.

The culminating of any celestial object, is the time when it *transits*, or comes to the meridian of any place.

The variation of the compass, is the deviation of the magnetic needle from the true north or south point; or

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The anomalistic month is the period in which the moon returns to its apogee, consisting of $27^d\ 13^h\ 18^m\ 34^s$.

The tropical year is the period in which the sun returns to the same point in the ecliptic, consisting of $365^d\ 5^h\ 48^m\ 5.6^s$.

The sidereal year is the period in which the sun returns to the same star, consisting of $365^d\ 6^h\ 9^m\ 10^s$.

The anomalistic year is the period in which the sun returns to the same apsis, consisting of $365^d\ 6^h\ 15^m\ 46^s$.

the difference between the true and magnetic amplitude, or azimuth, of any of the heavenly bodies.

The hour angle, is an angle at the pole of the equator, contained between the meridian of any place and the meridian which passes through the sun or a star.

Diurnal and nocturnal arcs, are such portions of the parallels of declination, above and below the horizon, as are described by any celestial body from the time of its rising to its setting, and from its setting to its rising (*e*).

The equation of time, is the difference between apparent or solar time, as shown by a true sun-dial, and true or mean time, as shown by a well-regulated clock or watch (*f*).

Consequentia is the apparent motion of a planet eastward, or according to the order of the signs; and *antecedentia* is its apparent motion westward, or contrary to the order of the signs.

(*e*) In places situated on the equator, the rational horizon cuts all the parallels of declination into two equal parts; and, in this case, the sun and stars are 12 hours above the horizon and 12 hours below it. But in places lying between the equator and the elevated pole, the parallels of declination are unequally divided, the greater arc being above the horizon, and the less below it. And in all cases between the equator and the depressed pole, the greater arc is below the horizon, and the less above it.

(*f*) The equation of time arises from three causes, viz. the obliquity of the ecliptic, the earth's unequal motion in its orbit, and the precession of the equinoxes; which variation is always equal to the difference between the sun's apparent increase of right ascension in 24^h and $3^m\ 56\frac{1}{2}^s$, the mean increase in that time.

The conjunctions, oppositions, &c. of the planets, and the characters by which they are denoted, are as follows:

☌ *conjunction*, is when two planets are referred to the same point of the ecliptic.

⚹ *sextile*, is when they are 2 signs, or 60° distant from each other.

□ *quartile*, is when they are 3 signs, or 90° distant from each other.

△ *trine*, is when they are 4 signs, or 120° distant from each other.

☍ *opposition*, is when they are 6 signs, or 180° distant from each other.

The conjunctions and oppositions are also called *the syzyges*, and the quartile aspects *the quadratures*, these terms being chiefly applied to the moon.

The nodes are the two points in which the orbits of the primary planets intersect the ecliptic; or the points in the orbits of the primary planets which are the intersections of the orbits of their secondaries, or satellites.

That point, or node, where the planet ascends from the south towards the north of the ecliptic, is called the north, or *ascending node;* and the other the south, or *descending node;* the two points being marked thus:

☊ Moon's north, or ascending node.
☋ Moon's south, or descending node.

The names and characters of the sun, and all the primary planets, yet known, are as follows:

☉ The sun, ☿ mercury, ♀ venus, ⊖ earth,
♂ mars, ♃ jupiter, ♄ saturn, ♅ uranus,
⚳ ceres, ⚴ pallas, ⚵ juno, ⚶ vesta (*g*).

The nutation of the earth's axis, is a periodical revolution of it, depending upon the place of the moon's ascending node; by which the situations of the stars are apparently changed.

The precession of the equinoxes is an uniform retrograde motion of the equinoctial points in the plane of the ecliptic, which affects the longitude, right ascension, and declination of the stars (*h*).

A constellation is a collection of stars, supposed to be circumscribed by the outlines of some assumed figure, as a ram, a serpent, an hercules, &c. which division was found necessary in order to direct an observer to that part of the heavens where any particular star is situated (*i*).

(*g*) The four last of these planets, which have been lately discovered, lie between Mars and Jupiter, and are too small to be seen with the naked eye.

(*h*) By comparing together the places of the fixed stars, as deduced from observation, astronomers have found that their longitudes increase about 50¼″ annually; which increase must necessarily cause an irregular motion in the same star with respect to the equator: hence, the right ascensions and declinations of the stars are constantly varying; so that some of those which had formerly north declination have now south declination, and the contrary. Their latitudes are also subject to a small variation, from the same cause.

(*i*) In order that the memory may not be overburthened by a

PROBLEM I.

To reduce the time, under any known meridian, to the corresponding time at Greenwich; and the converse.

RULE.

Convert the longitude of the place into time, by reckoning 15 degrees to an hour.

Then, this time, added to the time from the preceding noon, at the given place, if it be west, or subtracted from it, if east, will give the Greenwich time.

Or, conversely,

To the time from the preceding noon at Greenwich, add the longitude of the place in time, if east, or subtract it, if west; and the sum or difference will be the corresponding time under the meridian of that place (*k*).

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multiplicity of names, astronomers mark the stars of each constellation with a letter of the Greek alphabet, denoting those which are the most conspicuous by α, the next by β, and so on in succession; by which means they may be easily spoken of and referred to, as occasion requires. Several of the brightest stars have also proper names, as Arcturus, Orion, Aldebaran, Antares, &c.

(*k*) Since the earth makes one revolution on its axis, from west to east, in 24 hours, the sun must *apparently* make one revolution round the earth, from east to west, in the same time. And as the longitude of all places on the earth is reckoned on the equator, the whole of this circle, which is divided into 360°, must pass the sun in 24 hours; therefore, every 15 degrees of motion is one hour in time, every degree 4 minutes, and every minute 4 seconds. Whence a place one degree eastward of Greenwich will have noon, and every hour of the day, 4 minutes sooner than at Greenwich; and a place one degree westward of Greenwich will have noon, and every hour of the day, 4 minutes later.

It may also be observed, that the astronomical day is supposed

Example 1.

It is required to find the time at Greenwich, when it is $5^h\ 15'$ P. M. at a place in longitude $74^\circ\ 21'$ w.

$$\begin{array}{r|l} 3 & 74^\circ\ 21' \\ \hline 5 & 24^\circ\ 47' \\ \hline & 4^h\ 57'\ 24'' \end{array}$$

Time at given place - - - $5^h\ 15'$
Longitude in time - - - $4^h\ 57'\ 24''$ w.
Time at Greenwich - - - $10^h\ 12'\ 24''$ P. M.

Example 2.

What is the time in longitude $62^\circ\ 9'$ w. when it is $15^h\ 11'\ 14''$ at Greenwich?

$$\begin{array}{r|l} 3 & 62^\circ\ \ 9' \\ \hline 5 & 20^\circ\ 43' \\ \hline & 4^h\ \ 8'\ 36'' \end{array}$$

Apparent time at Greenwich - $15^h\ 11'\ 14''$
Longitude in time - - - - - - $4^h\ \ 8'\ 36''$ w.
Time at the given place - - - $11^h\ \ 2'\ 38''$

Example 3.

It is required to find the time at Greenwich answering to $3^h\ 45'\ 39''$ of May 1st in lon. $110^\circ\ 2'\ 15''$ east.

Ans. April 30th at $20^h\ 25'\ 30''$.

Example 4.

It is required to find the time at a place in longitude

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to begin at noon, or 12 hours later than the civil day of the same denomination; and is counted up to 24 hours, or the succeeding noon, when the next day begins: so that, for instance, January 10th, at 15 hours, is the same as January 11th at 3 in the morning, by the civil reckoning.

107° 48′ east, which corresponds with $4^h\ 51'\ 30''$ Greenwich time. Ans. $12^h\ 2'\ 42''$.

PROBLEM II.

To reduce the declination of the sun, as given in the Nautical Almanac for Greenwich, to any other meridian, and to any given time of the day.

RULE.

Convert the longitude of the place (if different from that of Greenwich) into time, as in the last problem.

Then, as 24 hours : change of the sun's declination in that time : : the time from noon : the proportional part required.

Which added to, or subtracted from, the declination at noon, according as it is increasing or decreasing, will give the declination at the time and place required.

Note. The sun's change of longitude, or of right ascension, may also be reduced to any given time and place, by a similar process (*l*).

Example 1.

Required the sun's declination August 9th 1808, at $4^h\ 46'$ A. M. in longitude 123° 7′ west.

$123°\ 7' = 8^h\ 12'\ 28''$, the time at which the inhabitants of Greenwich have the sun *before* those in longitude 123° 7′ w. Hence, when it is $4^h\ 46'$ in the morning at the latter place, it is $12^h\ 58'\ 28''$ at Greenwich, or 58′ 28″ P. M.

(*l*) The sun's longitude, right ascension in time, and declination, are given in the Nautical Almanac for every day in the year, at noon.

Sun's declin. at noon Aug. 9th 1808,	15° 51′ 58″ N.
Ditto - - - - - - - - - - - 10th - - -	15° 34′ 32″ N.
Decrease of declination in 24 hours	0° 17′ 26″

As 24^h : 17′ 26″ : : 58′ 28″ : 42.46″

Sun's declin. at noon Aug. 9th 1808,	15° 51′ 58″
Decrease of declination in 58′ 28″ -	42.46″
Sun's true declin. at given place -	15° 51′ 15½″

Example 2.

Required the sun's declination on the 17th of May 1808, at 11^h 48′ Greenwich time. Ans. 19° 27′ 39″.

Example 3.

Required the sun's declination on Feb. 20th 1808, at 2^h 15′ P. M. in lon. 134° 56′ east. Ans. 11° 19′ 4″.

Example 4.

What was the sun's right ascension at Greenwich, October 18th 1808, at 7^h 40′ P. M.? Ans. 13^h 33′ 35″.

Example 5.

What was the sun's right ascension at noon on the 27th of April 1808, in longitude 86° 45′ west?

Ans. 2^h 19′ 35″.

Example 6.

What was the sun's longitude, January 19th 1808, at 4^h 35′ in longitude 21° 15′ east of Greenwich?

PROBLEM III.

To reduce the declination of the moon, as given in the Nautical Almanac, for Greenwich, to any given time under a known meridian.

RULE.

Reduce the given time (if necessary) to the meridian of Greenwich, by Prob. I; and find the variation of declination in 12 hours by the almanac.

Then, as 12 hours : this variation : : the interval between the reduced time and the preceding noon or midnight : the proportional part required.

Which part being added to, or subtracted from, the moon's declination at the preceding noon or midnight, according as it is increasing or decreasing, will give the declination at the time required.

Note. The moon's right ascension, semidiameter, and horizontal parallax, may also be reduced to any time and place by a similar process (*m*). See Prob. XXII.

Example 1.

Required the declination of the moon at Greenwich on the 13th of August 1808, at $8^h\ 15'\ 53''$ P. M. apparent time.

Moon's declin. at noon 13th Aug. 1808, $15^\circ\ 37'$ N.
Ditto at midnight - - - - - - - - - - - - $16^\circ\ 53'$
Increase of declination in 12 hours - - $1^\circ\ 16'$

$12^h : 1^\circ\ 16' :: 8^h\ 15'\ 53'' : 52'\ 20\tfrac{1}{2}''$

At noon - - - $15^\circ\ 37'$
Increase - - - $52'\ 20\tfrac{1}{2}''$
Declin. required $16^\circ\ 29'\ 20\tfrac{1}{2}''$

(*m*) The moon's right ascension, declination, semidiameter, horizontal parallax, and time of passing the meridian at Greenwich, are given in the Nautical Almanac for every day in the year, at noon and midnight.

Example 2.

Required the moon's declination Sept. 20th 1808, at 6^{h} 13′ apparent time, under the meridian of Greenwich. Ans. 4° 30′ 10″ s.

Example 3.

Required the moon's declination Dec. 17th 1808, at 15^{h} 9′ apparent time, in longitude 64° 14′ w.

Ans. 19° 1′ 59″.

Example 4.

Required the moon's horizontal parallax and semidiameter, December 7th 1808, at 10^{h} 15′, in longitude 39° 40′ east.

Ans. { ☾'s Parallax - - 56′ 6.7″
☾'s Semidiam. - 15′ 17″

PROBLEM IV.

To find the culminating of the stars, or the times of their coming to the meridian.

RULE.

Subtract the sun's right ascension for the given day from the right ascension of the star (increased by 24 hours, if necessary), and the remainder will be the time of the star's culminating nearly.

Then, as 24 hours, added to the increase or decrease of the sun's right ascension in that time : 24 hours : : the time of the star's culminating nearly : the true time of culminating, at Greenwich.

And if the time of culminating be required for any other meridian than that of Greenwich, add the longitude in time to the true time of culminating at Greenwich, if the place be west, or take their difference, if it be east, and the result will give the time of culminating at the given place (*n*).

Example 1.

At what time did the star Arcturus come to the meridian of Greenwich on the 19th of Dec. 1808?

Arcturus's right ascen. (1808) + 24^h - $38^h\ 6'\ 54''$
Sun's right ascension - - - - - - - - - - - $17^h\ 49'\ 6''$
Time of star's culminating nearly - - - $20^h\ 17'\ 48''$

Sun's right ascen. at noon Dec. 19, 1808, $17°\ 49'\ 6''$
Ditto - - - - - - - - - - - Dec. 20, 1808, $17°\ 53'\ 32.7''$
Sun's increase of right ascen. in 24 hours $4'\ 26.7''$

Then, $24^h\ 4'\ 26.7'' : 24^h :: 20^h\ 17'\ 48'' : 20^h\ 14'\ 3''$ true time of Arcturus's culminating at Greenwich, or $8^h\ 14'\ 3''$ December 20th 1808.

Example 2.

At what time did Aldebaran culminate at Greenwich on the 23d of November 1808, his right ascension being $4^h\ 24'\ 54.8''$? Ans. $12^h\ 26'\ 57\frac{1}{2}''$ at night.

Example 3.

At what time did Regulus culminate at Greenwich on the 9th of February 1808, his right ascension being $9^h\ 58'\ 8''$? Ans. $27'\ 48''$ past 12 at night.

(*n*) If to any given time there be added the sun's right ascension for that time, the sum (rejecting 24, if necessary) is the right ascension of the mid heaven, which, being sought for in a table of the right ascensions, will show what star is on, or near, the meridian at that time.

Example 4.

At what time, on the 21st of February 1808, was Sirius, or the dog-star, on the meridian of a place whose longitude is 166° 30′ east? Ans. 9ʰ 12′ 14½″.

PROBLEM V.

To find the time when the moon, or a planet, will culminate, or pass the meridian.

RULE.

From the planet's motion in right ascension in 24 hours, (increased by 24 hours, if necessary,) take that of the sun, if the planet be progressive; or add them together, if it be retrograde.

Then, as 24 hours, diminished by this sum or difference, when the planet's motion is greater than the sun's, or increased by it when the sun's apparent motion is the greater : 24 hours : : the planet's right ascension at noon, (+24 hours, if necessary,) diminished by the sun's : the time of its transit at Greenwich.

And if the time of culminating be required for any other meridian than that of Greenwich, it may be obtained by the following rule:

As 24 hours, added to the daily change of the moon or planet in right ascension : the longitude of the place, converted into time : : the daily change above mentioned : the reduction required. Which added to, or subtracted from, the time of the moon's passing the meridian of Greenwich, according as the longitude is west or east, will give the apparent time of the transit at the required place.

And if the longitude in time be added to, or subtracted from, this, it will give the time at Greenwich (*o*).

Example 1.

Required the time of the moon's culminating at Greenwich, lat. 51° 28′ 40″ N. on the 17th Aug. 1808.

Sun's right ascen. at noon 17th Aug. 1808,	$9^h\ 46'\ 30.4''$
Ditto - - - - - - - 18th - - - -	$9^h\ 50'\ 14.2''$
Increase of motion in 24 hours - - -	$3'\ 43.8''$

Moon's right ascen. at noon 17th Aug. 1808,	89° 20′
Ditto - - - - - - - - 18th - - - -	102° 45′
Increase of the moon's motion in 24 hours	13° 25′

From 13° 25′ = 53′ 40″ of time
take 3′ 43.8″
leaves 49′ 56.2″

Which is the excess of the moon's motion above the sun's in 24 hours.

Moon's right ascension 89° 20′ = $5^h\ 57'\ 20''$
24^h
$29^h\ 57'\ 20''$
Sun's right ascension - - - - $9^h\ 46'\ 30.4''$
Difference - - - - - - - - - - $20^h\ 10'\ 49.6''$

Then, as 24 hours − 49′ 56.2″ (23^h 10′ 3.8″) : 24 hours : : 20^h 10′ 49″ : 20^h 54′ 19″ the true time of the moon's passing the meridian of Greenwich, or 8^h 54′ 19.4″ A. M. Aug. 18th 1808.

(*o*) The right ascensions of the planets are not given in the Nautical Almanac; but they may be readily computed from their geocentric longitudes and latitudes (which are to be found in that work), and the obliquity of the ecliptic.

Example 2.

It is required to find the time of the moon's passing the meridian of Greenwich on the 21st of July 1808.

Ans. $11^h\ 3'\ 1.6''$ A. M. July 22d 1808.

Example 3.

Required the time at Greenwich when the moon's centre passed the meridian of a place in longitude 21° west, on the 14th of November 1808.

Ans. $10^h\ 54'\ 50.9''$ A. M. Nov. 15th 1808.

Example 4.

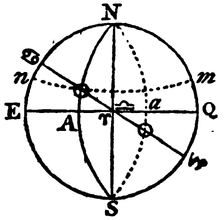
It is required to find the time when the planet Mercury passed the meridian at Greenwich on the 22d of December 1808, his right ascension being $19^h\ 4\frac{1}{2}'$.

PROBLEM VI.

The sun's declination, and the obliquity of the ecliptic, being given, to find his longitude and right ascension.

Example 1.

On the 23d of April 1808, the sun's declination was $12°\ 33'\ 58''$, and the obliquity of the ecliptic $23°\ 27'\ 42''$; required his longitude and right ascension (*p*).



(*p*) The construction of the figures in all the following examples is left for the exercise of the learner.

1. *To find* ⊙*'s longitude.*

As sin ∠ ⊙ ♈ A - -	23° 27′ 42″ -	9.6000308
Is to sin dec. ⊙ A -	12° 33′ 58″ -	9.3375910
So is rad, or sin - -	90° - - - - -	10.0000000
To sin ⊙'s lon. ♈ ⊙	33° 7′ 28″ -	9.7375602

2. *To find* ⊙*'s right ascension.*

As rad, or sin - - -	90° - - - - -	10.0000000
Is to tan dec. ⊙ A -	12° 33′ 58″ -	9.3481208
So is cot ∠ ⊙ ♈ A -	23° 27′ 42″ -	10.3624926
To sin right asc. ♈ A	30° 54′ 10.8″	9.7106134

And if this be converted into time, at the rate of 15° to an hour, the right ascension will be 2^h 3′ 36.7″

Note. As the sun's declination is the same at equal distances from the equinoctial points, it is necessary to know the time of the year, or what part of the ecliptic he was in when an observation was made, in order to determine his longitude and right ascension, which are reckoned from the first point of aries quite round the globe.

Thus, while the sun is moving from ♈ towards ♋, or is in the 1st quadrant of the ecliptic, the longitude is ♈ ⊙, the declin. ⊙ A, and the right ascen. ♈ A.

When he has passed the solstice ♋, and is descending towards ♎, he is then in the 2d quadrant, and his longitude, or distance from ♈, must be taken from 180°, in which case the remainder ♎ ⊙ becomes the hypothenuse, and the declination is still north; but the arc A ♎ is the supplement of the right ascension, and must, therefore, be taken from 180°.

When he has passed the point ♎, and is descending towards ♑, he has got into the 3d quadrant; in which case the excess above 180°, or his distance from ♎, will be the hypothenuse ♎ ☉; the declination will be south, and the arc ♎ A must be added to 180°, for the right ascension, estimated from ♈.

When he has passed the solstice ♑, and is ascending towards ♈, he is then in the 4th quadrant; in which case the longitude must be taken from 360° to give the hypothenuse ♎ ☉; the declination is south, and the arc ♎ A must be taken from 360° to give the right ascension from ♈.

Example 2.

On the 1st of Jan. 1808, the obliquity of the ecliptic was 23° 27′ 42″, and the sun's declination 23° 5′ 8″ s., and increasing; required his longitude and right ascension.

Ans. { ☉'s long. - - - 279° 59′ 9½″
☉'s right asc. 280° 51′ 50½″

Example 3.

On the 1st July 1808, the obliquity of the ecliptic was 23° 27′ 40½″, and the sun's right ascension in time 6^h 40′ 48.3″; required his longitude and declination.

Ans. { ☉'s long. 359° 22′ 24.7″
☉'s decl. 23° 7′ 46.6″ N.

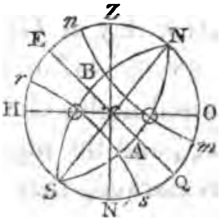
Any two of the four things mentioned in this problem being given, the rest may be found by one of the cases for right-angled spherical triangles.

PROBLEM VII.

The latitude of the place, and the sun's declination, being given, to find his amplitude, ascensional difference, and the time of his rising and setting.

Example 1.

Given the latitude of Greenwich 51° 28′ 40″ N., and the sun's declination on the 21st of June 1808, being the longest day, 23° 27′ 41″ N.; required his amplitude, ascensional difference, and the time of his rising and setting.



1. *To find the sun's amplitude.*

:	Sin colat. ∠ A ♈ ⊙	38° 31′ 20″	9.7943612
:	Sin declin. A ⊙ - -	23° 27′ 41″	9.6000259
: :	Rad, or sin - - - -	90° - - - -	10.0000000
:	Sin ⊙'s ampl. ♈ ⊙	39° 44′ 6.9″	9.8056647

Which is the amplitude from the east or west point of the horizon; and its complement 50° 15′ 53.1″ shows how far from the north the sun rises or sets on the longest day, at Greenwich.

2. *To find the sun's ascensional difference.*

:	Rad, or sin - - - -	90° - - - -	10.0000000
:	Tan lat. ∠ N ♈ ⊙	51° 28′ 40″	10.0990491
: :	Tan declin. ⊙ A -	23° 27′ 41″	9.6375010
:	Sin asc. diff. ♈ A -	33° 2′ 16.2″	9.7365501

Where if 33° 2′ 16.2″ be converted into time, at the rate of 15° to an hour, it gives $2^h\ 12'\ 9''$ for the time the sun rises before, and sets after 6 o'clock, on the longest day.

Hence $6^h - 2^h\ 12'\ 9'' = 3^h\ 47'\ 51''$ time of sun rising:
and $6^h + 2^h\ 12'\ 9'' = 8^h\ 12'\ 9''$ time of sun setting.

Also, if the latter of these be doubled, it gives $16^h\ 24'\ 18''$ for the length of the longest day at Greenwich; and if the former be doubled, it gives $7^h\ 35'\ 42''$ for the length of the night.

But on the shortest day at Greenwich, which is when the sun has 23° 27′ 41″ of s. declination, the lengths of the days and nights change place, the day being then $7^h\ 35'\ 42''$, and the night $16^h\ 24'\ 18''$.

It may also be remarked, that if *r s* be a parallel of declination as far to the south as *n m* is to the north, the hour-circle N B S, passing through ☉, the place of the sun, at its rising or setting, will form a △ ♈ ☉ B = △ ♈ ☉ A, where the amplitude ♈ ☉ is to the southward of the east and west points.

From which it is evident, that when the latitude and declination have the same name, the sun rises before and sets after 6; but when they are of contrary names, the sun rises after and sets before 6.

When the sun's declination is equal to, or greater than, the co-latitude of the place, (which can only happen to places upon or within the polar circles) the parallel of declination *nm* will not cut the horizon H O, and consequently the sun will neither rise nor set at these times. And the same will hold with respect to those stars whose co-declination, or polar distance N ☉, is equal to, or less than, the latitude of the place, or the elevation of the pole N O, and in the same hemisphere (q).

Example 2.

Given the sun's amplitude 39° 44′ 6.9″, and his declination 23° 27′ 41″ N., to find the latitude of the place, and the time of the sun's rising and setting.

$$\text{Ans.}\begin{cases}\text{latitude} & 51^\circ\ 28'\ 40''\ \text{N.}\\ \odot\ \text{rises} & 3^{h}\ 47'\ 51''\\ \odot\ \text{sets} & 8^{h}\ 12'\ \ 9''\end{cases}$$

Example 3.

Given the latitude of the place 51° 28′ 40″ N., and the sun's amplitude 39° 44′ 6.9″ N. of the east, required his declination, ascensional difference, and time of rising and setting.

$$\text{Ans.}\begin{cases}\text{declin.}\ - & 23^\circ\ 27'\ 41''\ \text{N.}\\ \text{asc. diff.} & 33^\circ\ \ 2'\ 16.2''\\ \odot\ \text{rises}\ - & 3^{h}\ 47'\ 51''\\ \odot\ \text{sets}\ - & 8^{h}\ 12'\ \ 9''\end{cases}$$

Note. Any two of the five things mentioned in this problem being given, the rest may be found by some of the cases in right-angled spherical triangles.

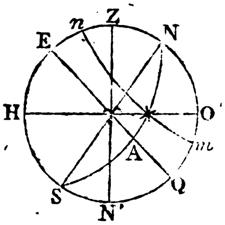
PROBLEM VIII.

The latitude of the place being given, and the declination of a star, to find its amplitude, ascensional difference, and the time of its rising and setting.

(q) When the latitude and declination have the same name, the difference between the right ascension and the ascensional difference is the oblique ascension; and their sum is the oblique descension. But when they are of contrary names, their sum is the oblique ascension, and their difference the oblique descension.

Example 1.

Required the amplitude, ascensional difference, and the time of rising and setting of Arcturus, at Greenwich, lat. 51° 28′ 40″ N. on December 19th 1808, its declination being 20° 11′ 21.8″.



1. *To find the amplitude* ♈ ✱.

:	Sin co-lat. ∠ A ♈ ✱	38° 31′ 20″	9.7943612
:	Rad, or sin - - - -	90° - - - -	10.0000000
: :	Sin declin. A ✱ - -	20° 11′ 21.8″	9.5379757
:	Sin amp. ♈ ✱ - -	33° 39′ 3.8″	9.7436145

Which amplitude is always of the same name as the declination; and since the variation of a star's declination is extremely small, the same star may be considered as having constantly the same amplitude during the whole year, in the same latitude.

2. *To find the ascensional difference* ♈ A.

:	Rad, or sin - - -	90° - - - -	10.0000000
:	Cot co-lat. A ♈ ✱	38° 31′ 20″	10.0990491
: :	Tan declin. A ✱ -	20° 11′ 21.8″	9.5655151
:	Sin asc. diff. A ♈ -	27°.30′ 39¼″	9.6645642

Which, converted into time, gives $1^h\ 50'\ 2.6''$

And $6^h + 1^h\ 50'\ 2.6'' = 7^h\ 50'\ 2.6'' =$ arc. A E, or half the time of the star's continuance above the horizon.

From time of *'s culmin^{g}. prob. IV. 8^{h} 14′ 3.4″ A.M.
Take - - - - - - - - - - - 7^{h} 50′ 2.6″
Time of *'s rising in morn^{g}. Dec. 20th 24′ 0.8″

To time of *'s culminating - - - 8^{h} 14′ 3.4″
Add - - - - - - - - - - - - 7^{h} 50′ 2.6″
Time of the *'s setting - - - - - 16^{h} 4′ 6″

Or 4^{h} 4′ 6″ in the afternoon Dec. 20th 1808.

Where it may be observed, that on account of the small change in the declination of the stars, the same star, in any latitude, may be considered as having the same ascensional difference throughout the year. And as the diurnal difference of the same star's rising, culminating, and setting, in the same latitude, is nearly equal to the diurnal difference of the sun's right ascension, which is 3′ 56½″, this may be taken for the daily difference of the rising, southing, and setting of any fixed star in the same latitude (*r*).

Example 2.

It is required to find at what time Sirius, or the dog-star, rose and set at a place in lat. 51° 28′ 40″ N., and long. 166° 30′ E., on the 21st of February 1808, its declination being 16° 27′ 29.9″ S.

Ans. { Sirius rises at 4^{h} 39′ 22½″ in the morning.
—— sets at 1^{h} 45′ 6½″ in the afternoon.

(*r*) The mode of solution made use of in this and the preceding problem may be also applied to the rising or setting of the moon, or a planet. But when great exactness is required, the declination of the sun or planet must be calculated as near to the time of rising and setting as possible, especially for the moon, on account of her

Example 3.

It is required to find the amplitude, and time of rising and setting of Aldebaran at Greenwich, on the 23d of Nov. 1808, its declination being 16° 6′ 47″ N.

Ans. { Amplitude 26° 27′ 45″ N.
Rises at 5h 1′ 51″ evening.
Sets at 7h 52′ 4″ next morning.

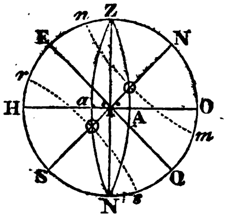
Note. Any two of the five things, mentioned in this problem, being given, the rest may be found, as in the former.

PROBLEM IX.

The latitude of the place, and the sun's declination, being given, to find his altitude and azimuth at 6 o'clock.

Example 1.

At Greenwich in latitude 51° 28′ 40″ N. on the longest day, when the sun's declination is 23° 27′ 41″ N., it is required to find his altitude and azimuth at 6 o'clock in the morning, or evening.



1. *To find the altitude* ⊙ A.

: Rad, or sin - - - - 90° - - - - 10.0000000
: Sin declin. ♈ ⊙ - 23° 27′ 41″ 9.6000259
: : Sin lat. ∠ ⊙ ♈ A - 51° 28′ 40″ 9.8934103
: Sin alt. ⊙ A - - - - 18° 8′ 55.3″ 9.4934362

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swift and irregular motion. Also, the declination of the sun near the equinoxes changes considerably in the compass of an hour.

2. *To find the azimuth* A ⊙.

:	Rad, or sin	90°	10.0000000
:	Cos lat. ⊙ ♈ A	51° 28′ 40″	9.7943612
::	Tan declin. ♈ ⊙	23° 27′ 41″	9.6375010
:	Cot azim. A ⊙	74° 52′ 25.9″	9.4318622

On the shortest day, at London, the parallel of s declination *r s* cuts the 6 o'clock hour-circle N S below the horizon; and as the △ˢ ⊙ ♈ A, ⊙ ♈ *a*, are equal in all their parts, the depression *a* ⊙ below the horizon, on the shortest day, at 6 o'clock, will be equal to the altitude ⊙ A, at the same hour on the longest day; and the azimuth will also be equal to it, if estimated from the south. So that on the 21st of June, at Greenwich, the sun will bear N. 74° 52′ 25.9″ E. at 6 o'clock in the morning, and N. 74° 52′ 25.9″ W. at 6 in the evening; but on the 21st of December, at the same hours, it will bear S. 74° 52′ 25.9″ E., and S. 74° 52′ 25.9″ W. (*s*).

Example 2.

Given the sun's declination 23° 27′ 41″ N. and his

(*s*) From a due consideration of this problem, it is evident, that as the declination increases, the altitude increases and the azimuth lessens; and the contrary, when the declination is diminishing. So that on the days of the equinoxes, on which the sun has no declination, his altitude at 6 o'clock will be nothing, or he will then be in the horizon; and the azimuth, in this case, being 90° from the north, the sun will be due east in the morning, and west in the evening; that is, on the days of the equinoxes, the sun rises and sets at 6 o'clock, in the east and west points of the horizon.

altitude at 6 o'clock in the morning 18° 8′ 55.3″, required his azimuth and the latitude of the place.

Ans. { Azimuth - - 74° 52′ 25.9″ from N.
Lat. of place 51° 28′ 40″ N.

Example 3.

Given the sun's declination 23° 27′ 41″ N., and his azimuth at 6 o'clock in the morning 74° 52′ 26″ from the N., required his altitude and the latitude of the place.

Ans. { Altitude - - 18° 8′ 55½″
Lat. of place 51° 28′ 40″ N.

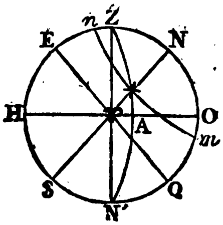
Any two of the four things, mentioned in this problem, being given, the rest may be found as before.

PROBLEM X.

The latitude of the place, and the declination of a star being given, to find at what time it will be upon the 6 o'clock hour-circle, and its altitude and azimuth at that time.

Example 1.

At what time, on the 19th of Dec. 1808, did the star Arcturus appear upon the 6 o'clock hour-circle, at Greenwich, lat. 51° 28′ 40″ N.; and what was its altitude and azimuth at that time, its declin. being 20° 11′ 21.8″ N.?



2. The time of the star's passing the meridian, as found by problem IV, will be $20^h\ 14'\ 3''$; and consequently it will be upon the 6 o'clock hour-circle at $(20^h\ 14'\ 3'' - 6^h)\ 14^h\ 14'\ 3''$ in the eastern hemisphere; and at $(20^h\ 14'\ 3'' + 6^h)\ 26^h\ 14'\ 3''$, in the western hemisphere.

2. *To find the altitude* ✱ A.

:	Rad, or sin - - - -	90° - - - -	10.0000000
:	Sin declin. ♈ ✱ -	20° 11′ 21.8″	9.5379757
: :	Sin lat. ✱ ♈ A - -	51° 28′ 40″	9.8934103
:	Sin alt. ✱ A - - - -	15° 39′ 54″	9.4313860

3. *To find the azimuth* A O.

:	Rad, or sin - - - -	90° - - - -	10.0000000
:	Tan declin. - - - -	20° 11′ 21.8″	9.5655151
: :	Cos lat. ✱ ♈ A - -	51° 28′ 40″	9.7943612
:	Cot azim. A O - - -	77° 6′ 1.7″	9.3598763

In which case, it may be observed, that on account of the small change in the right ascension and declination of a star, it may, without material error, be said to have the same altitude and azimuth every time it arrives at the 6 o'clock hour-circle; and the difference of the times it arrives there may be considered as equal to the diurnal difference of the sun's right ascension.

Example 2.

At what time did Aldebaran appear upon the 6 o'clock hour-circle at Greenwich, lat. 51° 28′ 40″ N., on the 23d of November 1808, and what was its altitude and azimuth at that time, its declination being 16° 6′ 47″ N.?

Ans. { In the evening at $6^h\ 26'\ 57\frac{1}{2}''$
✱'s altitude - - - 12° 32′ 26.8″
✱'s azimuth - - 79° 48′ 1″ from N.

Example 3.

At what time did Regulus appear upon the 6 o'clock hour-circle at Greenwich, lat. 51° 28′ 40″ N., on the 9th of February 1808, and what was its altitude and azimuth at that time, its declination being 12° 54′ 7″?

Ans. { In the evening at 6h 27′ 48″
*'s altitude - - 10° 3′ 38″
*'s azimuth - - 81° 52′ 50″ N.

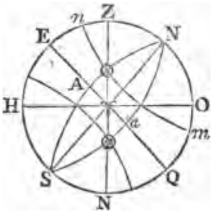
Note. Any two of the five things, mentioned in this problem, being given, the rest may be found as before.

PROBLEM XI.

The latitude of the place, and the sun's declination, being given, to find his altitude and the time when he will be due east or west.

Example 1.

In latitude 51° 28′ 40″ N., when the sun's declination is 19° 47′ 27″, it is required to find his altitude, and the time when he will appear upon the prime vertical, or due east or west.



1. *To find the altitude* ♈ ☉.

:	Sin lat. A ♈ ☉ - -	51° 28′ 40″	9.8934103
:	Sin declin. ☉ A - -	19° 47′ 27″	9.5296707
: :	Rad, or sin - - - -	90° - - - -	10.0000000
:	Sin alt. ♈ ☉ - - -	25° 38′ 37″	9.6362604

2. *To find the hour from 6, ♈ A.*

:	Rad, or sin - - - -	90° - - - -	10.0000000
:	Tan declin. ⊙ A -	19° 47′ 27″	9.5561111
::	Cot lat. ∠ A ♈ ⊙ -	51° 28′ 40″	9.9009509
:	Sin ♈ A - - - - - - -	16° 38′ 46″	9.4570620

Which converted into time, gives 1ʰ 6′ 35″ for the time from 6 o'clock.

Hence, the sun will be exactly east at 7ʰ 6′ 35″ in the morning, or west at 4ʰ 53′ 25″ in the afternoon (*t*).

Example 2.

The sun's declination being 19° 47′ 27″ N. and his altitude, when upon the prime vertical, 25° 38′ 37″, it is required to find the latitude of the place, and the hour of the day.

Ans. { Lat. 51° 28′ 40″ N.
Time 7ʰ 6′ 35″ morning
Or 4ʰ 53′ 25″ afternoon.

Example 3.

In latitude 51° 28′ 40″ N. the sun's altitude, when

(*t*) From this problem it appears, that when the latitude of the place and the sun's declination have the same name, the altitude and time from six o'clock increase as the latitude and declination increase; and having contrary names, the same thing happens, with this difference, that in the former case the days lengthen, on account of the increase of the latitude and declination, whereas, in the latter they shorten on that account. When the latitude of the place is less than the declination, the sun never appears on the prime vertical.

If this problem be worked, in the △ ♈ ⊙ A, for the longest day, at Greenwich, it will give 4ʰ 39′ 16″ for the time before and after noon when the sun is due east and west; and in the △ ♈ ⊙ *a*, it will give the time for the shortest day.

on the prime vertical, was 25° 38′ 37.3″, required his declination and the hour of the day.

Ans. { Declin. 19° 47′ 27″ N.
Time - 7^h 6′ 35″ morning
Or - 4^h 53′ 25″ afternoon.

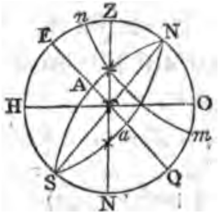
Note. Any two of the four things, mentioned in this problem, being given, the rest may be found as before.

PROBLEM XII.

The latitude of the place being given, and the declination of a star, to find its altitude, and the time when it will be due east, or west.

Example 1.

At what time did Arcturus appear due east or west from Greenwich, on the 19th of December 1808; and what was its altitude at that time, its right ascension being 14^h 6′ 25″, and declination 20° 11′ 21.8″ N.?



1. The star culminates at 20^h 14′ 34″, as found by prob. IV.

1. *To find the altitude* ♈ *.

:	Sin lat. A ♈ * - -	51° 28′ 40″	9.8934103
:	Rad, or sin - - - -	90° - - - -	10.0000000
: :	Sin declin. * A - -	20° 11′ 21.8″	9.5379757
:	Sin alt. ♈ * - - -	26° 10′ 33″	9.6445654

2. *To find the hour from 6,* ♈ A.

;	Rad, or sin	90°	10.0000000
:	Tan declin. ✱ A	20° 11′ 21.8″	9.5655151
: :	Cot lat. ∠ ✱ ♈ A	51° 28′ 40″	9.9009509
:	Sin ♈ A	17° 1′ 17.2″	9.4664660

And if 72° 58′ 42.8″ (the complement of ♈ A) be converted into time, it gives 4^{h} 51′ 54.8″, which subtracted from the time of the star's southing, leaves 15^{h} 22′ 8½″ or 3^{h} 22′ 8½″ the next morning, (December 20th), when the star appeared due east, and added, gives 25^{h} 5′ 58″, or 1^{h} 5′ 58″ P. M. (December 20th,) when the star appeared due west (*u*).

Example 2.

At what time did Aldebaran appear due east or west at Greenwich, lat. 51° 28′ 40″ N., on the 23d November 1808, and what was its altitude at that time, its declination being 16° 6′ 47″ N.?

Ans. { Altitude 20° 46′ 38½″
East at - 7^{h} 20′ 8½″ evening
West at 5^{h} 33′ 46½″ next morning.

Example 3.

At what time did Regulus appear due east or west at Greenwich, lat. 51° 28′ 40″ N., on the 9th of Fe-

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(*u*) The height of the same star upon the prime vertical, in any place, is always nearly the same, for the reasons already assigned; and the difference of the times of its coming to the prime vertical will be equal to the difference of the times of its culminating, which is nearly equal to the diurnal difference of the sun's right ascension.

bruary 1808; and what was its altitude at that time, its declination being 12° 54′ 7″ N.?

$$\text{Ans.}\begin{cases}\text{Altitude } 16^\circ\ 34'\ 57'' \\ \text{East - - } 7^h\ 9'\ 49\frac{1}{2}''\text{ evening} \\ \text{West - - } 5^h\ 45'\ 46''\text{ next morning.}\end{cases}$$

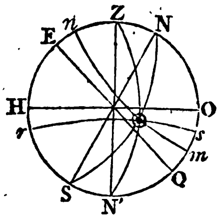
Note. Any two of the things, mentioned in this problem, being given, the rest may be found as before.

PROBLEM XIII.

The sun's declination, and the latitude of the place, being given, to find the time of day-break in the morning, and the end of twilight in the evening.

Example 1.

Given the latitude of the place 51° 28′ 40″ N., and the sun's declination 9° 48′ 20″ N., to find the time of day-break in the morning, and the end of twilight in the evening.



The crepusculum circle *r s* being 18° below the horizon, we shall have

Sun's polar dist. O N = 80° 11′ 40″
Sun's zenith dist. O Z = 108°
Comp^t. of lat. - N Z = 38° 31′ 20″
2|226° 43′ = sum
113° 21′ 30″ = ½ sum.

113° 21′ 30″	113° 21′ 30″	113° 21′ 30″
108°	80° 11′ 40″	38° 31′ 20″
5° 21′ 30″	33° 9′ 50″	74° 50′ 10″

Log sine - - - -	113° 21′ 30″	9.9628631
Log sine - - - -	5° 21′ 30″	8.9702738
Sum of log sines - - - - - -		18.9331369
Its complement - - - - -		1.0668631
Log sine - - - -	33° 9′ 50″	9.7380157
Log sine - - - -	74° 50′ 10″	9.9846090
		2|20.7894878
Tan ½ ∠ZN☉ - -	68° 3′ 10″	10.3947439
	2	
	136° 6′ 20″ ∠ZN☉.	

And if this be converted into time, it gives 9h 4′ 25″ = time from noon when the ☉ is 18° below the horizon.

Hence the day breaks at 2h 55′ 34½″ in the morning, and twilight ends at 9h 4′ 25″ in the evening, supposing the sun's declination to undergo no change during that time (*x*).

(*x*) When the declination becomes greater than the difference between the co-latitude and 18°, the parallel of declination *n m* will not cut the parallel *r s*, and consequently there will then be no night at that place, the twilight continuing from sun-setting to sun-rising; which takes place at London from 22d May to about the 21st of July.

It may also be observed, that as the sun sets more obliquely at some times of the year than at others, it follows that he will be longer in ascending or descending through an arc of 18° below the horizon at one season than at another. When he is on the same side of the equator as the visible pole, the duration of twilight will constantly increase till he enters the tropic, at which

Example 2.

Given the latitude of the place $51^\circ\ 28'\ 40''$ N., and the sun's declination $9^\circ\ 48'\ 20''$ S., to find the time of day-break in the morning, and the end of twilight in the evening.

Ans. { Day breaks at - $4^h\ 53'\ 25\frac{1}{2}''$ morning.
Twilight ends at $7^h\ 6'\ 34''$ evening.

Example 3.

Given the latitude of the place $51^\circ\ 28'\ 40''$ N., and the sun's declination on the shortest day $23^\circ\ 27'\ 36''$ S., to find the time of day-break in the morning, and the end of twilight in the evening.

Ans. { Day breaks at - $6^h\ 0'\ 59''$ morning.
Twilight ends at $5^h\ 59'\ 1''$ evening.

PROBLEM XIV.

The latitude of the place, and the sun's declination

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time it will be the longest. It will then decrease till some time after he passes the equinox, but will increase again before he enters the other tropic; whence, there must be some point between the tropics, where the duration of twilight is the shortest; which point may be found by the following analogy:

As rad : tan 9° (half the distance of the crepusculum circle from the horizon) : : sin lat. of the place : sin sun's declination, when the twilight is the shortest.

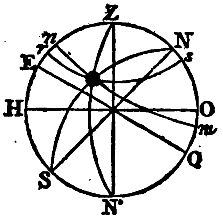
Which declination is always of a contrary name with the latitude.

Note. At Greenwich, lat. $51^\circ\ 28'\ 40''$ N. the time of shortest twilight is when the sun has $7^\circ\ 7'\ 25''$ S. declination, answering to March 2d and October 11th; between which days it increases, while from the latter to the former it decreases; its whole duration being $1^h\ 56'\ 32''$. For a demonstration of the above analogy, see Emerson's Miscellanies, p. 492, and Vince's Astronomy, vol. i.

and altitude, being given, to find his azimuth and the hour of the day.

Example 1.

In latitude 51° 28′ 40″ N., the sun's true altitude was found to be 46° 23′ when his declination was 23° 27′ 41″ N.; what was his azimuth, and the hour of day, when the observation was made?



Co-lat. - - Z N = 38° 31′ 20″
Co-alt. - - Z ⊙ = 43° 37′
Co-declin. N ⊙ = 66° 32′ 19″

2|148° 40′ 39″ = sum
74° 20′ 19½″ = ½ sum.

74° 20′ 19½″	74° 20′ 19½″	74° 20′ 19½″
38° 31′ 20″	66° 32′ 19″	43° 37′
35° 48′ 59½″	7° 48′ 0½″	30° 43′ 19½″

Log sine - - - -	74° 20′ 19½″	9.9835697
Log sine - - - -	7° 48′ 0.5″	9.1326373
Sum of log sines - - - - - -		19.1162070
Complement - - - - - - - -		0.8837930
Log sine - - - -	35° 48′ 59½″	9.7672981
Log sine - - - -	30° 43′ 19½″	9.7083140
		2|20.3594051
Tan ½ ∠ N Z ⊙ - -	56° 31′ 46″	10.1797025
	2	
	113° 3′ 32″ = azim. from N.	

Log sine	74° 20′ 19½″	9.9835697
Log sine	30° 43′ 19½″	9.7083140
Sum of log sines		19.6918837
Complement		0.3081163
Log sine	7° 48′ 0.5″	9.1326373
Log sine	35° 48′ 59½″	9.7672981
		2|19.2080517
Tan ½ ∠ ☉ N Z	21° 53′ 27½″	9.6040258
	2	
	43° 46′ 55″ = hour ∠ from noon.	

Which converted into time, gives 2h 55′ 7½″.

Hence, the observation was made either at 9h 4′ 52″ in the morning, or at 2h 55′ 7½″ in the afternoon (y).

Example 2.

Given the latitude of the place, 51° 28′ 40″ N. the sun's declination, 19° 34′ 26″ N. and the altitude of his centre, 38° 20′; required the azimuth and the hour from noon.

Ans. { Azimuth - - S. 72° 5′ 28½″ E.
Hour from noon 3h 29′ 33½″.

(y) If the declination and latitude are of contrary names, the things required may be found by the same mode of operation, except that the side N ☉ being, in this case, greater than 90°, the declination must be added to 90°, instead of subtracted from it, as in the above example.

Note. The observed altitude of the sun's upper or lower limb, must be corrected for refraction, parallax, and the dip of the horizon, in order to obtain the true altitude of his centre, which is that to be used. See problem XXII.

Example 3.

In latitude 48° 52′ s., the sun's declination was 19° 34′ 26″ N., and his true altitude 20° 30′, required the hour of the day. Ans. 0ʰ 54′ 12½″ from noon.

Example 4.

At Greenwich, in latitude 51° 28′ 40″ N., in the afternoon, when the sun's declination was 9° 39′ 40″, and the altitude of his centre 24° 59′ 50″, what was his azimuth from the north, and the hour of the day?

Ans. { Azimuth 152° 0′ 18½″
Hour - - 1ʰ 42′ 15½″ P. M.

Example 5.

In latitude 39° 55′ N., longitude 35° 50′ W., the altitude of the sun's lower limb, on the 7th of May 1808, at 5ʰ 30′ 35″ P. M., per watch was 15ʰ 40′ 53″; how much was the watch too fast or too slow (z)?

Ans. Watch too slow 1′ 54.7″.

Note. Any three of the five things, mentioned in this problem, being given, the rest may be found by some of the cases of oblique-angled triangles.

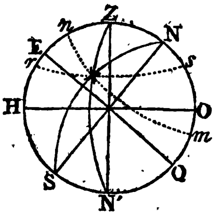
PROBLEM XV.

Given the time of the year, the latitude of the place, and the altitude of a known fixed star, to find the hour of the night when the observation was made.

(z) In the practical application of problems of this kind, it will be proper to take several altitudes of the sun or star, and the corresponding times, per watch, within one or two minutes of time of each other, and to use the mean of these observations instead of any one of them, taken singly, as the errors arising from the imperfection of the instrument, &c. will, in this case, be rendered almost insensible.

Example 1.

Some time in the night on the 19th of December 1808, in the latitude of Greenwich, 51° 28′ 40″ N., the altitude of Arcturus was observed to be 26° 10′ 30″; required the hour of the night.



Co-lat. - - Z N 38° 31′ 20″
Co-alt. - - Z ✱ 63° 49′ 30″
Co-declin. N ✱ 69° 48′ 38″
2|172° 9′ 28″

86° 4′ 44″	86° 4′ 44″	86° 4′ 44″
38° 31′ 20″	63° 49′ 30″	69° 48′ 38″
47° 33′ 24″	22° 15′ 14″	16° 16′ 6″

Log sin - - - - - -	86° 4′ 44″	9.9989822
Log sin - - - - - -	22° 15′ 14″	9.5783088
Sum of log sines - - - - - -		19.5772910
Complement - - - - - - - -		0.4227090
Log sin - - - - - -	47° 33′ 24″	9.8680242
Log sin - - - - - -	16° 16′ 6″	9.4473686
		2|19.7381018
Tan ½ ∠ Z N ✱ - -	36° 29′ 34″	9.8690509
	2	
	72° 59′ 8″ hour from noon.	

Which converted into time, gives 4^{h} 51' 56½" for the time before the star came to the meridian.

Time of *'s culmin^{g}. at Greenwich (prob. IV.) - - - -	20^{h} 14' 34" Dec. 19th.
Timebefore * passes the meridian	4^{h} 51' 56½"
Hour of the night - - - - -	15^{h} 22' 37½"

Or 3^{h} 22' 37½" A. M. Dec. 20th 1808.

Example 2.

In latitude 48° 55' 40" N. longitude 65° 59' 20" W. on the 14th of April 1808, the altitude of Aldebaran, when west of the meridian, was 22° 20' 20"; required the apparent time of observation.

Ans. Hour of the night, 12^{h} 14' 45".

Example 3.

In latitude 10° 26' S. and longitude 166° 30' E. on the 21st of February 1808, the altitude of Sirius, when east of the meridian, was observed to be 21° 10' 50"; required the true time of the night when the observation was made.

Ans. February 21st at 4^{h} 28' 59½".

Example 4.

On the 30th of January 1808, in latitude 53° 24' N. and longitude 25° 18' W. the mean of several altitudes of Procyon to the west, and of Alphacca, or α Coronæ, to the east of the meridian, were observed separately, by two persons, at the same instant of time, the mean of which was 16^{h} 38' 38", the mean altitude of Procyon $19^\circ\ 51'\ 18''$, and the mean altitude of α Coronæ $42^\circ\ 8'$; from which it is required to find the error of the watch.

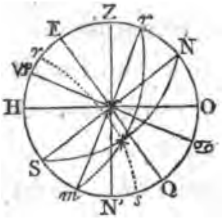
Ans. Hour at night for Procyon $16^h\ 36'\ 37''$
Ditto - - - - for Alphacca $16^h\ 35'\ 27''$
Watch too fast - - - - $2'\ 35''$

PROBLEM XVI.

The right ascension and declination of a fixed star, or planet, being given, to find its latitude and longitude.

Example 1.

Required the right ascension and declination of Aldebaran in Taurus, its latitude being $5^\circ\ 28'$ s. its longitude $2^s\ 6^\circ\ 56'$, and the obliquity of the ecliptic $23^\circ\ 28'$ (*a*).



1. *To find the co-declination* s ✱.

Here sm - - $= 23^\circ\ 28'$ the obliquity of ecliptic
$\angle\, {*}\, m\, s = 156^\circ\ 56'$ star's longitude + 90°
$m\, {*}$ - - $= 84^\circ\ 32'$ compᵗ. of star's latitude.

(*a*) A method of resolving this problem, which is better adapted for obtaining accurate results, in certain cases, is given by Maskelyne, in his Introduction to Taylor's Logarithms. The problem itself may be varied so as to admit of several cases; but except that given above, and its converse, none of these are of any great practical use.

Whence, by case VIII. of oblique $\angle^d$ spherical $\triangle^s$;

:	Rad, or sin	90°	10.0000000
:	Cos $\angle * m$ s	156° 56′	9.9638112
::	Tan $m *$	84° 32′	11.0190794
:	Tan arc ϕ	95° 57′	10.9828906
:	Cos arc ϕ	95° 57′	9.0156135
			0.9843865
:	Cos $m *$	84° 32′	8.9789408
::	Cos (s$m \sim \phi$)	72° 29′	9.4785423
:	Cos s $*$	106° 57′	9.4418696
		90°	
	Declination	16° 57′ s.	

2. To find the co-right ascension $*$ s*m*.

:	Sin s $*$	106° 57′	9.9807120
			0.0192880
:	Sin $\angle * m$ s	156° 56′	9.5930666
::	Sin $m *$	84° 32′	9.9980202
:	Sin $\angle *$ sm	24° 6′	9.6103748
		90°	
	The right ascension	65° 54′	

Example 2.

Required the latitude and longitude of Spica, or α Virginis, its right ascension being 13ʰ 15′ 5½″, its declination 10° 9′ 7½″ s. and the obliquity of the ecliptic 23° 27′ 42″.

Ans. { Latitude - - - 2° 2′ 17″ s.
Longitude 6ˢ 21° 9′ 41″

Example 3.

The latitude of the moon being 5° 6′ 37″ N. her longitude 10^s 21° 36½′, and the obliquity of the ecliptic 23° 27′ 42″; it is required to find her right ascension and declination.

Ans. { Right ascen. 322° 19′ 42″
Declination 9° 28′ 11″

Example 4.

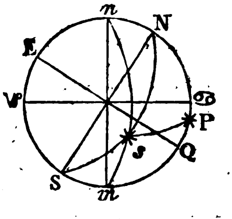
Required the right ascension of the planet Mercury in time, on the 31st of January 1808, its geocentric latitude being 2° s. and its geocentric longitude 10^s 4° 57′. Ans. 20^h 31′ 17″.

PROBLEM XVII.

The right ascensions and declinations of two stars, or their latitudes and longitudes, being given, to find their distance.

Example 1.

It is required to find the distance between Sirius in Canis Major, and Procyon in Canis Minor, the right ascension of the former being 6^h 36′ 41″, and its declination 16° 27′ 30″ s. and the right ascension of the latter 7^h 29′ 14″, and its declination 5° 42′ 34″ N.



Here { ∠ $*$ s P = 13° 8′ 15″ the diff. of right ascen. $*^s$
s $*$ = 73° 32′ 30″ compᵗ. of Sirius's declin.
s P = 95° 42′ 34″ Procyon's declin. + 90°.

Whence, by case VIII. of oblique-angled spherical △s,

:	Rad, or sin - - - -	90° - - - -	10.0000000
:	Cos ∠ * s p - - -	13° 8′ 15″	9.9884820
: :	Tan s * - - - - -	73° 32′ 30″	10.5295562
:	Tan arc φ - - - -	73° 7′ 27″	10.5180382
:	Cos arc φ - - - - -	73° 7′ 27″	9.4628447
			0.5371552
:	Cos s * - - - - -	73° 32′ 30″	9.4522742
: :	Cos (s p ∼ φ) - - -	22° 35′ 6″	9.9653471
:	Cos * p - - - - -	25° 41′ 45″	9.9547765

Where * p is the distance of the two stars required.

Example 2.

It is required to find the distance between Rigel in Orion, and Procyon in Canis Major, the right ascension of the former being 5^h 55′ 19″, and its declination 8° 25′ 49″ s.; and the right ascension of the latter 7^h 29′ 14″, and its declination 5° 42′ 34″ N.

Ans. dist. 27° 21′ 3½″.

Example 3.

Required the distance between Sirius and Spica Virginis, the latitude of the former being 39° 33′ 22″ s. and its longitude 3^s 11° 26′ 10″; and the latitude of the latter 2° 2′ 17″ s. and its longitude 6^s 21° 9′ 49″.

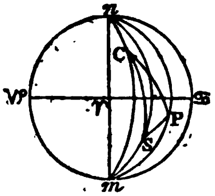
Ans. dist. 96° 10′ 18″.

PROBLEM XVIII.

The places of two stars being given, and their distances from a third star, or comet, to find the place of the third object.

Example 1.

Suppose the distance of a comet, or new star, as measured by a sextant, to be 65° 47′ from Sirius, whose latitude is 39° 33′ s. and longitude 3^s 11° 13′, and 51° 6′ from Procyon, whose latitude is 15° 58′ s. and longitude 3^s 22° 55′; it is required to find the latitude and longitude of this comet or star.



1. In the △ s *n* p, we have given

s *n* = 129° 33′ dist. of Sirius from pole of eclip.
p *n* = 105° 58′ do. of Procyon from same pole.
∠ s *n* p = 11° 42′ diff. longitude of the two stars.

Hence, by case VIII. of oblique $\angle^d$ spherical $\triangle^s$,

:	Rad, or sin	90°	10.0000000
:	Cos ∠ s *n* p	11° 44′	9.9908291
: :	Tan s *n*	129° 33′	10.0831235
:	Tan arc φ	130° 8′	10.0739526
:	Cos arc φ	130° 8′	9.8092691
			0.1907309
:	Cos s *n*	129° 33′	9.8039699
: :	Cos (p *n* ∼ φ)	24° 10′	9.9601655
:	Cos s p	25° 42′	9.9548663
:	Sin s p	25° 42′	9.6371484
			0.3628516
:	∠ s *n* p	11° 44′	9.3082590
: :	Sin p *n*	105° 58′	9.9829140
:	Sin ∠ *n* s p	26° 48′	9.6540246

2. In the △ CSP, we have given

SP = 25° 42′ dist. of Sirius and Procyon
CS = 65° 47′ dist. of comet and Sirius
CP = 51° 6′ dist. of comet and Procyon.

Hence, by case XI. of oblique ∠d spherical △s,

25° 42′
65° 47′
51° 6′
2|142° 35′

71° 17′	71° 17′	71° 17′
65° 47′	51° 6′	25° 42′
5° 30′	20° 11′	45° 35′

Log sin - - -	71° 17′ - -	9.9764036
Log sin - - -	20° 11′ - -	9.5378508
		19.5142544
		0.4857456
Log sin - - -	5° 30′ - -	8.9815729
Log sin - - -	45° 35′ - -	9.8538619
		2|19.3211804
Tan - - - - -	24° 36′ - -	9.6605902
	2	
	49° 12′ ∠ CSR	
	26° 48′ ∠ nSP	
	22° 24′ ∠ CSn	

3. In the △ nSC, we have given

Sn = 129° 33′ Sirius's dist. from pole of eclip.
SC = 65° 47′ dist. of Sirius and comet
∠ CSn = 22° 24′ found as above.

Hence, by case VIII. of oblique $\angle^d$ spherical $\triangle^s$,

:	Rad, or sin	90°	10.0000000
:	Cos ∠ csn	22° 24′	9.9659285
: :	Tan sc	65° 47′	10.3470119
:	Tan arc φ	64° 3′	10.3129404
:	Cos arc φ	64° 3′	9.6410640
			0.3589360
:	Cos sc	65° 47′	9.6129833
: :	Cos (sn∼φ)	65° 30′	9.6177270
:	Cos cn	67° 8′	9.5896463
		90°	
		22° 52′ lat. of comet.	
:.	Sin cn	67° 8′	9.9644537
			0.0355463
:	Sin ∠ csn	22° 24′	9.5810052
: :	Sin sc	65° 47′	9.9599952
: .	Sin ∠ cns	22° 10′	9.5765467
		101° 13′ long. of Sirius	
		22° 10′ diff. long.	
		79° 3′ long. of comet.	

Note. The same things may also be determined from the right ascensions and declinations of the two stars, by referring them to the equator instead of the ecliptic.

Example 2.

Suppose an unknown star was found to be 66° 10′ 12″ distant from Capella, whose declination is 45° 47′ 15″ N., and its right ascension $5^h\ 2'\ 31''$; and that its distance was 25° 15′ 18″ from Procyon, whose declination is 5° 42′ 34″ N. and right ascension $7^h\ 29'\ 14''$; required the declination and right ascension of the unknown star.

Ans. { Declination - - 12° 6′ 23″
Right ascension 9ʰ 8′ 14″

Example 3.

Suppose the distance of an unknown star, or planet, was observed to be 67° 45′ 24″ from Sirius, whose right ascension is 99° 10′ 15″, and declination 16° 27′ 30″ s. and to be 51° 4′ 53″ distant from Procyon, whose right ascension is 112° 18′ 30″, and declination 5° 42′ 34″ N.; it is required to find the right ascension and declination of this unknown star or planet (*b*).

Ans. { Declination - - 49° 17′ 36″ N.
Right ascension 80° 43′ 15″.

PROBLEM XIX.

The day of the month, the sun's declination, and the latitude of the place, being given, to find the apparent time of his centre appearing in the horizon.

Example 1.

Given the sun's declination, at noon, on the 21st of June 1808, 23° 27′ 41″, and the latitude of the place

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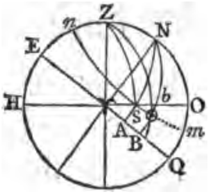
(*b*) By the assistance of the above problem, and a knowledge of the true situation of any particular star, the places of all the rest may be determined; it being chiefly by this means that astronomers have rectified the places of the fixed stars, and thence, by a similar mode of proceeding, found the true places of the planets. In the same manner, we may also find the distance between any two places on the surface of the earth.

51° 28′ 40″ N; required the apparent time of his centre appearing in the horizon (*c*).



Here, the example being given for the longest day, the sun's declination may be considered the same at his rising as at noon; the variation not being more than 5″ in 24 hours at this time, though near the equinoxes it varies above 1′ in an hour.

Hence, by the tables, his horizontal refraction being = 33′, his parallax 9″, and his semidiameter 15′ 47″, if S be the point where the upper limb of the sun rises, and *b* that of its apparent rising, we shall have $33' + 15'\,47'' - 9'' = b\odot$ the distance of his centre below the horizon; and consequently,

$Z\odot = 90^\circ\,48'\,36.8''$ app. dist. of ⊙'s centre from zen.
$N\odot = 66^\circ\,32'\,19''$ dist. of ⊙'s centre from N. pole
$ZN = 38^\circ\,31'\,20''$ the complement of the latitude.

(*c*) The apparent time of the rising and setting of the heavenly bodies always differs from the true time, on account of their being elevated by refraction, and depressed by parallax. The sun's horizontal parallax is about 9″; and therefore his upper limb will appear in the horizon when it is 32′ 51″ below it; and as his semidiameter is then 15′ 47″, his centre will appear in the horizon when it is 48′ 38″ below it: but a star, having no sensible parallax, will appear in the horizon when it is 33′ below it, or 90° 33′ from the zenith.

Hence, by case XI. of oblique-angled spherical △s,

90° 48′ 36.8″
66° 32′ 19″
38° 31′ 20″
2|195° 52′ 15.8″

97° 56′ 7.9″	97° 56′ 7.9″	97° 56′ 7.9″
66° 32′ 19″	90° 48′ 36.8″	38° 31′ 20″
31° 23′ 48.9″	7° 7′ 31″	59° 24′ 47.9″

Log sin - - - - - -	97° 56′ 7.9″	9.9958212
Log sin - - - - - -	7° 7′ 31.1″	9.0935608
		19.0893820
		0.9106179
Log sin - - - - - -	31° 23′ 48.9″	9.7168075
Log sin - - - - - -	59° 24′ 47.9″	9.9349326
		2|20.5623581
Log tan - - - - -	62° 22′ 23.2″	10.2811790
	2	
	124° 44′ 46½″ ∠ ZNO.	

Which, converted into time, gives 8^h 18′ 59″ = the true time from noon, when the sun's centre appears in the horizon:

Hence, his apparent central rising is 3^h 41′ 1″, and his setting 8^h 18′ 59″ (*d*).

The true time of the sun's rising and setting on this day, has before been shown to be 3^h 47′ 51″, and

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(*d*) As refraction is the means of occasioning an error in the rising and setting of all the celestial objects, the same cause will necessarily produce an error in their amplitudes, as may be seen by comparing the triangles ♈ A S and ♈ B *b*; but this may be avoided, by making the altitude of ☉'s lower limb = 16′ + dip of the horizon.

8^h 12′ 9″, hence the apparent day is 13′ 40″ longer than the astronomical day.

Example 2.

Required the apparent rising and setting of the sun's centre, in latitude 51° 28′ 40″ N. supposing his declination to be 23° 27′ 36″ S.

Ans. 8^h 5′ 22″ and 3^h 54′ 38″.

Example 3.

Required the apparent time of the central rising and setting of the sun, in latitude 51° 28′ 40″ N. when his declination is 19° 47′ 20″ N. supposing it to undergo no variation from sun-rise to sun-set.

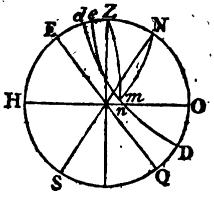
Ans. 4^h 6′ 15″ and 7^h 53′ 44½″.

PROBLEM XX.

The latitude and longitude of a place, and the day of the month, being given, to find the time of the moon's rising and setting at that place.

Example 1.

Required the time of the rising and setting of the moon at Greenwich, latitude 51° 28′ 40″ N. on the 18th of August 1808 (*e*).



(*e*) The horizontal parallax of the moon, being from 53′ to 62′, always exceeds the horizontal refraction; therefore, when the moon's upper limb appears in the horizon she is really above it, by a quantity equal to the horizontal parallax, minus the refraction, Let n be the place of the moon, when in the horizon, m the point where she becomes visible, and e her place on the meridian; in which case, de will represent her change of declination from the time of her rising to that of her transit.

Then, by calculating as in problem v. the true time of her passing over the meridian will be 8h 54′ 19″ in the morning. And by prob. iii. her declination, when on the meridian, will be found to be 18° 56′ 51½″ n., and her horizontal parallax 56′ 31.8″; also, by the tables, the horizontal refraction = 33′.

Hence, in the △ zNm, we have given

$$\begin{cases} zm = 90^\circ + 33' - 56'\ 31.8'' & = 89^\circ\ 36'\ 28.6'' \\ Nm = 90^\circ - 18^\circ\ 56'\ 51.7'' & = 71^\circ\ 3'\ 8.2'' \\ zN = 90^\circ - 51^\circ\ 28'\ 40'' & = 38^\circ\ 31'\ 20'' \end{cases}$$

Therefore, by case xi. of oblique-angled spherical △'s,

$$\begin{array}{r} 71^\circ\ 3'\ 8.2'' \\ 89^\circ\ 36'\ 28.6'' \\ 38^\circ\ 31'\ 20'' \\ \hline 2|199^\circ\ 10'\ 56.9'' \end{array}$$

$$\begin{array}{r} 99^\circ\ 35'\ 28.4'' \\ 71^\circ\ 3'\ 8.2'' \\ \hline 28^\circ\ 32'\ 20.2'' \end{array} \qquad \begin{array}{r} 99^\circ\ 35'\ 28.4'' \\ 38^\circ\ 31'\ 20'' \\ \hline 61^\circ\ 4'\ 8.4'' \end{array} \qquad \begin{array}{r} 99^\circ\ 35'\ 28.4'' \\ 89^\circ\ 36'\ 28.6'' \\ \hline 9^\circ\ 58'\ 59.8'' \end{array}$$

$$\begin{array}{lrr} \text{Log sin} \cdots\cdots & 99^\circ\ 35'\ 28.4'' & 9.9938863 \\ \text{Log sin} \cdots\cdots & 9^\circ\ 58'\ 59.8'' & 9.2389425 \\ \cline{3-3} & & 19.2328288 \\ \cline{3-3} & & 0.7671711 \\ \text{Log sin} \cdots\cdots & 28^\circ\ 32'\ 20.2'' & 9.6792062 \\ \text{Log sin} \cdots\cdots & 61^\circ\ 4'\ 8.4'' & 9.9421088 \\ \cline{3-3} & & 2|20.3884861 \\ \cline{3-3} \text{Log tan} \cdots\cdots & 57^\circ\ 22'\ 21.9'' & 10.1942430 \\ & 2 & \\ \cline{2-2} & 114^\circ\ 48'\ 43.8'' & \angle\, zNm. \end{array}$$

Which converted into time, gives 7^h 39′ 14.9″ = nearly the time the moon rises before she comes to the meridian. Whence 8^h 54′ 19.4″ − 7^h 39′ 14.9″ = 1^h 15′ 4.4″ = estimated time of the moon's rising.

Then, by calculating as in problem III. the moon's declination at *nearly* the time of her rising, will be found = 19° 26′ 12.2″ N., and her horizontal parallax 56′ 17.2″.

Hence, in the △ Z N *m*, we have, again, given

$$\left\{\begin{array}{ll} \mathrm{Z}\,m = 90^\circ + 33' - 56'\ 17.2'' & = 89^\circ\ 36'\ 42.7'' \\ \mathrm{N}\,m = 90^\circ - 19^\circ\ 26'\ 12.2'' & = 70^\circ\ 33'\ 47.7'' \\ \mathrm{Z}\ n = 90^\circ - 51^\circ\ 28'\ 40'' & = 38^\circ\ 31'\ 20'' \end{array}\right.$$

$$\begin{array}{r} 89^\circ\ 36'\ 42.7'' \\ 70^\circ\ 33'\ 47.7'' \\ 38^\circ\ 31'\ 20'' \\ \hline 2\,|\,198^\circ\ 41'\ 50.4'' \\ \hline \end{array}$$

$$\begin{array}{r} 99^\circ\ 20'\ 55.2'' \\ 70^\circ\ 33'\ 47.2'' \\ \hline 28^\circ\ 47'\ \ 7.4'' \\ \hline \end{array} \qquad \begin{array}{r} 99^\circ\ 20'\ 55.2'' \\ 89^\circ\ 36'\ 42'' \\ \hline 9^\circ\ 44'\ 12.5'' \\ \hline \end{array} \qquad \begin{array}{r} 99^\circ\ 20'\ 55.2'' \\ 38^\circ\ 31'\ 20'' \\ \hline 60^\circ\ 49'\ 35.2'' \\ \hline \end{array}$$

$$\begin{array}{lrr} \text{Log sin - - - - - -} & 99^\circ\ 20'\ 55.2'' & 9.9941930 \\ \text{Log sin - - - - - -} & 9^\circ\ 44'\ 12.5'' & 9.2282019 \\ \cline{3-3} & & 19.2223950 \\ \cline{3-3} & & 0.7776049 \\ \text{Log sin - - - - - -} & 28^\circ\ 47'\ \ 7.4'' & 9.6826237 \\ \text{Log sin - - - - - -} & 60^\circ\ 49'\ 35.2'' & 9.9410874 \\ \cline{3-3} & & 2\,|\,20.4013162 \\ \text{Log tan - - - - - -} & 57^\circ\ 47'\ 20.3'' & 10.2006581 \\ \cline{3-3} & 2 & \\ \cline{2-2} & 115^\circ\ 34'\ 40.7'' \ \angle\,\mathrm{Z\,N}\,m. & \end{array}$$

Which, converted into time, gives 7^h 42′ 18.7″ for the time the moon rises before she comes to the meridian.

Whence, 8ʰ 54′ 19.4″ − 7ʰ 42′ 18.7″ = 1ʰ 12′ 0.6″ the time from noon when the moon rises; which differs only 3′ 3.7″ from the time found above.

And if still greater exactness be required, the moon's declination may be found for 1ʰ 12′ 0.6″, and the operation repeated as before.

It is also evident that 8ʰ 54′ 19.4″ + 7ʰ 42′ 18.7″ = 16ʰ 36′ 38.1″ = nearly the time of her setting; but in order to obtain this time more rigorously, the variation of declination must be allowed for as in the preceding part of the problem.

Example 2.

Required the time of the moon's rising at Greenwich, latitude 51° 28′ 40″ N. on the 22d of July 1808.

Ans. 3ʰ 24′ 34½″ apparent time in the morning.

Example 3.

Required the time of the moon's rising and setting at Paris, latitude 48° 50′ 14″ N. and longitude 2° 20′ E., on the 25th of September 1808.

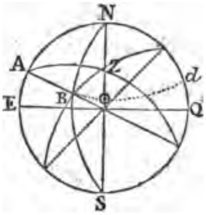
Ans. Rises 32′ 34½″ afternoon, apparent time at Paris; and sets at 9ʰ 10′ 55½″ in the evening.

PROBLEM XXI.

The sun's declination, two altitudes, and the time between the observations, being given, to find the latitude of the place.

Example 1.

At a place in the northern hemisphere, the sun's declination being 19° 39′ 12″ N., the true altitude of his centre, in the forenoon, was found to be 38° 20′ 30″, and at the end of an hour and a half afterwards 50° 26′ 10″; required the latitude of the place.



Here, A being the place of the sun at the time of the 1st observation, and B his place at the time of the 2d, the polar distances NA, NB will be equal, supposing the declination to remain the same for the whole of the interval.

Hence, in the isosceles △ NAB, there is given

NA = 70° 20′ 48″ the sun's polar dist. at 1st obser^n.
NB = 70° 20′ 48″ ditto - - - - - - - - at 2d obser^n.
∠ ANB = 22° 30′ the measure of elapsed time.

1. *To find the side* AB.

:	Rad, or sin - - - -	90° - - - -	10.0000000
:	Sin NA or NB - -	70° 20′ 48″	9.9739332
::	Sin $\frac{1}{2}$∠ ANB - - -	11° 15′ - - -	9.2902357
:	Sin $\frac{1}{2}$ AB - - - - - -	10° 35′ 12.5″	9.2641689
		2	
		21° 10′ 25″ AB.	

2. *To find the angle* ABN.

:	Cot $\frac{1}{2}$∠ ANB - - -	11° 15′ - -	10.7013382
:	Rad, or sin - - - -	90° - - - -	10.0000000
::	Cos NA or NB - -	70° 20′ 48″	9.5267634
:	Cot ABN - - - - - -	86° 10′ 21.4″	8.8254252

II. In the oblique-angled △ ABZ, there is given

AZ = 51° 39′ 30″ complement of 1st altitude
BZ = 39° 33′ 50″ complement of 2d altitude
AB = 21° 10′ 25″ as before found.

Whence, by case XI. of oblique-$\angle^d$ spherical $\triangle^s$,

51° 39′ 30″
39° 33′ 50″
21° 10′ 25″
2|112° 23′ 45″

56° 11′ 52″	56° 11′ 52″	56° 11′ 52″
39° 33′ 50″	51° 39′ 30″	21° 10′ 25″
16° 38′ 2″	4° 32′ 22″	35° 1′ 27″

Log sin - - - - -	56° 11′ 52″	9.9195823
Log sin - - - - -	4° 32′ 22″	8.8984402
		18.8180225
		1.1819774
Log sin - - - - -	16° 38′ 2″	9.4567571
Log sin - - - - -	35° 1′ 27″	9.7588541
		2|20.3975887
Tan - - - - - - -	57° 40′ 47″	10.1987943
	2	
	115° 21′ 35″ ∠ABZ	
	86° 10′ 21″ ∠ABN	
	29° 11′ 14″ ∠NBZ.	

III. In the triangle BZN, there is given

NB = 70° 20′ 48″ sun's polar distance
BZ = 39° 33′ 50″ complement of the 2d alt.
∠NBZ = 29° 11′ 14″ included ∠.

Whence, by case VIII. of oblique-angled spherical $\triangle^s$,

:	Rad, or sin - - - -	90° - - - -	10.0000000
:	Cos ∠NBZ - - - -	29° 11′ 14″	9.9410292
: :	Tan BZ - - - - - - -	39° 33′ 50″	9.9170909
:	Tan arc φ - - - -	35° 48′ 11″	9.8581201

: Cos arc φ - - - - - -	35° 48′ 11″	9.9090376
		0.0909623
: Cos B Z - - - - - -	39° 33′ 50″	9.8870064
: : Cos (φ ~ N B) - - -	34° 82′ 36″	9.9157669
: Cos N Z - - - - - - -	38° 28′ 5½″	9.8937357
	90°	
	51° 31′ 54″ lat. required(*f*).	

Example 2.

On a day, in the northern hemisphere, when the sun's declination was 20° 13′ 30″ N. his true altitude in the forenoon was observed to be 18° 32′ 5″, and three hours afterwards it was 39° 58′ 12″; from which it is required to find the latitude of the place.

Ans. 60° 10′ 24″ N.

Example 3.

When the sun's declination was 22° 37′ N. his correct altitude at 10^h 53′ 13″ in the forenoon was 53° 28′ 9″, and at 1^h 17′ 16″ in the afternoon it was 52° 47′ 29″; required the latitude of the place, supposing it to be north. Ans. 57° 3′ 42½″ N.

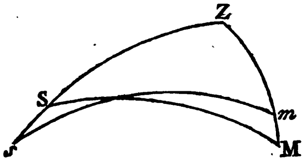
Example 4.

At a place in the northern hemisphere, when the sun's declination was 23° 28′ 36″ N. his corrected altitude, at 8^h 53′ 48″ in the forenoon, was 48° 42′ 20″, and at 9^h 46′ 9″ it was 55° 47′ 48″; required the true latitude. Ans. 49° 59′ 18½″ N.

(*f*) The principal use to which this problem is applied, is in questions of a similar nature to that given above; but several other things may be determined from the same data: such as the hour from noon when each altitude was taken, the azimuth at each observation, &c.

PROBLEM XXII.

The apparent distance of the moon from the sun, or a star, and their apparent altitudes, or zenith distances, being given, to find their true distances, as seen from the earth's centre (*g*).



Let Z M be the observed zenith distance of the moon, and Z *m* her true zenith distance, M *m* being the difference between the moon's refraction and her parallax in altitude.

Also let Z S be the observed zenith distance of the sun, or a star, and Z *s* its true zenith distance; S *s*

(*g*) Since the observed altitude of any celestial object is affected by refraction and parallax, the effects of which are always produced in a vertical direction, it is obvious that the observed distance between any two bodies will also be affected by the same causes. With regard to the fixed stars, the parallax vanishes, so that their places are only changed by refraction; but in observations of the moon particularly, the effect of parallax is very sensible, on account of her proximity to the earth: for which reasons, the true distance between the moon and any celestial object is, for the most part, considerably different from the observed distance.

It may also be remarked, that since the refraction of the sun, at the same altitude, is always greater than his parallax, his true place will be lower than his apparent place; and because the moon's parallax, at any given altitude, is always greater than the refraction at that altitude, her true place will be higher than her apparent place.

being the difference between the sun's refraction and parallax, or the refraction of a star.

Then, since in the triangle Z S M, the three sides Z S, Z M, and S M, are given, the vertical angle S Z M may be found, by case XI. of oblique-angled spherical triangles.

And, because in the triangle *z s m*, the two sides *z s*, *z m*, and the included angle *s z m*, are also known, the true distance *s m* may be found by case VIII. of oblique-angled spherical triangles.

But as this method, though direct and obvious, requires three separate statings, or analogies, for obtaining the true distance, it may be rendered more commodious in practice by incorporating the analytical formulæ for finding the angle Z and the side *s m* into a single expression; which, when converted into logarithms, gives the following rule, using the altitudes instead of the zenith distances.

RULE.

1. Take the difference of the apparent altitudes of the moon and star, or moon and sun, and half the difference of their true altitudes.

2. Also, take half the sum and half the difference of the apparent distance and the difference of the apparent altitudes.

3. To the log sines of this half sum and half difference, add the log cosines of the true altitudes, and the complements of the log cosines of the apparent altitudes, and take half the sum.

4. From this half sum take the log sine of half the difference of the true altitudes, and look for the remainder among the log tangents; which being found, take out the corresponding log cosine, without taking out the arc, which is unnecessary.

5. Subtract this log cosine from the log sine of half the difference of the true altitudes, above found, increased by 10 in the index; and the remainder will be the log sine of half the true distance (*h*).

Example 1.

Given	Apparent dist. of ☽ and ✱	51° 28′ 30″
	Apparent alt. of ☽'s centre	12° 30′ 4″
	Apparent alt. of ✱ - - - -	24° 48′ 17″
	True altitude of ☽'s centre	13° 20′ 40″
	True altitude of ✱ - - - -	24° 46′ 14″

Required the true distance of the moon and star.

24° 48′ 17″	51° 28′ 30″	
12° 30′ 4″	12° 18′ 13″	
12° 18′ 13″	2|63° 46′ 43″	
24° 46′ 14″	31° 53′ 21½″	- ½ sum
13° 20′ 40″	2|39° 10′ 17″	
2|11° 25′ 34″	19° 35′ 8½″	- ½ diff.
5° 42′ 47″ -		

(*h*) The method of reducing the apparent to the true distance, or of clearing it of the effects of refraction and parallax, being the most tedious part of the calculation for ascertaining the longitude, by the common spherical analogies, various compendious formulæ have been proposed, at different times, for the more easy solution of this problem; among the best of which are those of Borda, la Caille, de Lambre, Messrs. Dunthorn, Lyons, Maskelyne, Witchel, and Mendoza.

Log sin - - - 31° 53′ 21½″ - 9.7228640
Log sin - - - 19° 35′ 8½″ - 9.5253252
Log cos - - - 13° 20′ 40″ - 9.9881129
Log cos - - - 24° 46′ 14″ - 9.9580824
C Log cos - - - - 12° 30′ 4″ - 0.0104203
C Log cos - - - 24° 48′ 17″ - 0.0420371
2|39.2468421
19.6234210
Log sin - - - 5° 42′ 47″ - 8.9980259
Log tan of an arc - - - - - - 10.6253951
Corresponding log cosine - - - - 9.3627458
Log sin - - - 25° 34′ 54″ - 9.6352801
2
51° 9′ 48″ true distance.

Example 2.

Given
Apparent dist. ☉ and ☽'s centres 90° 21′ 17″
Apparent alt. of ☽'s centre - - - 5° 17′ 9″
Apparent alt. of ☉'s centre - - - 84° 7′ 20″
True altitude of ☽'s centre - - - 6° 9′ 14″
True altitude of ☉'s centre - - - 84° 7′ 15″

Required the true distance of the sun and moon's centres. Ans. 89° 29′ 18″.

Example 3.

Given
Apparent dist. ☉ and ☽'s centres 38° 45′ 38″
Apparent alt. of ☽'s centre - - - 29° 31′ 10″
Apparent alt. of ☉'s centre - - - 35° 43′ 6″
True altitude of ☽'s centre - - - 30° 19′ 43″
True altitude of ☉'s centre - - - 35° 41′ 54″

Required the true distance of the sun and moon's centres. Ans. 38° 28′ 20″.

PROBLEM XXIII.

The observed altitudes of the sun and moon, or of the moon and a star (*i*), and their apparent distance,

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(*i*) The principal stars used in finding the longitude are the following; which have been chosen on account of their lying near the moon's path:

1. *α Arietis*, a small star lying without the zodiac, about 22° to the right hand of the Pleiades.

2. *Aldebaran*, in the Bull's eye, a large conspicuous star, lying about half way between the Pleiades and the star which forms the western shoulder of Orion.

3. *α Pegasi*, a star lying something less than 45° to the right of *α* Arietis, being nearly in a line with this latter star and the Pleiades.

4. *Pollux*, a star a little to the northward of Aldebaran; being the left-hand one of two stars lying near together, called Castor and Pollux.

5. *Regulus*, a star about 38° S. E. of Pollux; being the southernmost of four bright stars to the N. E. of Aldebaran, forming a zigzag line.

6. *Spica*, or *α Virginis*, a white sparkling star, about 54° S. E. of Regulus.

7. *Antares*, a star lying to the right-hand of Regulus, and about 45° from Spica Virginis.

8. *Fomahault*, a star lying about 45° to the south of *α* Pegasi.

9. *α Aquila*, a star about 47° to the westward, or to the right-hand of *α* Pegasi.

These stars may be readily known, by finding them on a common celestial globe; or by means of their calculated distances, which are given in the Nautical Almanac, for every 3 hours of apparent time at Greenwich. For, the sextant being fixed to the distance between the moon and the star which ought to be observed, and the moon found upon the horizon-glass, it is only necessary to look to the east or west of the moon, according as the distance corresponds to the 8th and 9th or 10th and 11th pages of the Nautical Almanac, guiding the sextant in a line with the moon's shortest axis.

being given, together with the time, and the longitude by account, to find the true altitude.

RULE.

I. Add the longitude by account, converted into time, to the time at the given place, if west, or subtract it from that time, if east, and the result will give the time at Greenwich nearly; which call *the reduced time.*

II. Take the moon's semidiameter, and horizontal parallax, from the Nautical Almanac, for the noon and midnight between which the reduced time falls, and find their differences.

Then, as 12 hours : the difference of the semidiameters at those times : : the reduced time : a fourth number; which being added to, or subtracted from, the preceding semidiameter, according as the tables are increasing or decreasing, will give a result, to which if the augmentation of the semidiameter (tab. IV.) be added, the sum will be *the moon's true semidiameter, at reduced time.*

Also, As 12 hours : the difference of the horizontal parallaxes at those times : : the reduced time : a fourth number; which being added to, or subtracted from, the preceding parallax, according as the tables are increasing or decreasing, will give *the moon's horizontal parallax, at reduced time.*

III. Add the difference between the moon's semidiameter, at reduced time, and the dip of the horizon, to the observed altitude of her lower limb, and the sum will give *the apparent altitude of the moon's centre.*

Then, to the cosine of the altitude, thus found, add the logarithm of the horizontal parallax in seconds, at reduced time, and the sum, abating 10 in the index, will be the logarithm of the moon's parallax in altitude, in seconds; from which take the refraction in altitude, and the result, added to the apparent altitude of the centre, will give *the true altitude of the moon's centre.*

IV. If the sun be used, add the difference between his semidiameter on the given day (Naut. Alm.) and the dip of the horizon (tab. II.) to the observed altitude of his lower limb, and the sum will give *the apparent altitude of his centre;* from which take the difference between his refraction in altitude and his parallax (tab. II. and IV.), and the remainder will be *the true altitude of the sun's centre.*

Or, if a star be used, take the dip of the horizon (tab. II.) from its observed altitude, and the remainder will be *its apparent altitude;* from which subtract the refraction, and the result will be *the star's true altitude.*

V. To the observed distance of the sun and moon's nearest limbs add their semidiameters, at reduced time, or subtract them for the furthest limbs, and the result will be *the apparent distance of the sun and moon's centres.*

Or, to the observed distance of a star from the moon's nearest limb add her semidiameter, at reduced time, or subtract it for the furthest limb, and the result will be *the apparent distance of the star from the moon's centre.*

VI. With the apparent altitudes, the true altitudes, and the apparent distance, find the true distance, by problem XXII. And if the watch has not been previously regulated, the true time must now be found from the altitude of the sun's centre, or a star, and the latitude of the place, as in problem XV. observing to proportion the sun's declination to the reduced time.

VII. Look in the Nautical Almanac, on the given month and day, for the computed distance between the moon and sun, or star, and if it be found exactly, the time at Greenwich will stand at the top of the column; but if not, find the nearest distances to it, both less and greater, and take their difference, and also the difference between the computed distance and the *earliest* Ephemeris distance.

Then, as the first difference : 3 hours : : the second difference : a fourth number; which being added to the time standing over the *earliest* Ephemeris distance, will give *the true time at Greenwich.*

And, if the difference between the time at the given place and the time at Greenwich, be converted into degrees, it will give the longitude required; which will be east or west, according as the time at the given place is greater or less than the time at Greenwich.

Example 1.

On the 30th of January 1796, in longitude 10° 46′ E. by account, at 10^h 15′ P. M. the distance of the moon's furthest limb from the star Regulus was 63° 50′ 20″, the altitude of the moon's lower limb 24° 18′ 40″, and the altitude of the star 45° 13′ 15″, the eye being 18 feet above the plane of the horizon; required the true longitude of the place.

I.

Time per watch - - - - - - -	10^{h} 15' 0" P. M.
Longitude 10° 46' east - - - - -	43' 40"
Reduced time - - - - - - -	9^{h} 31' 20" P. M.

II.

☽'s semid^{r}. at noon - -	15' 3"	Horizontal parallax - -	55' 14"
Do. at midnight - - - -	14' 59"	Ditto - - - - - - - - - -	54' 59"
First diff. - -	0' 4"	Second diff. - -	0' 15"
12^{h} : 4" : : 9^{h} 31' 20" :	0' 3"	12^{h} : 15" : : 9^{h} 31' 20" :	0' 12"
☽'s semid^{r}. at noon - - -	15' 3"	Horizontal par. at noon	55' 14"
☽'s semid^{r}. at red. time -	15' 0"	Ditto at reduced time -	55' 2"
☽'s augmentation - - -	0' 7"		60
☽'s true semid^{r}. at red. time - - - - - -	15' 7"	In seconds - - - -	3302

III.

☽'s observed alt. - -	24° 18' 40"	Cos ☽'s app. alt. - -	9.9590383
Semidiam. 15' 7" Dip - - - 4' 3"	0° 11' 4"	Hor. paral. 3302 log -	3.5187771
		Par. in alt. 3005 log	3.4778154
App. alt. of ☽'s cent.	24° 29' 44"	3005 sec.	50' 5"
Correction - - -	48' 1"	☽'s refrac.	2' 4"
True alt. of ☽'s cen.	25° 17' 45"	Correction	48' 1"

IV.

✱'s observed alt. - -	45° 13' 15"
Dip of horizon - -	4' 3"
✱'s apparent alt. -	45° 9' 12"
Refraction - - -	57"
✱'s true alt. - - -	45° 8' 15"

V.

Obs. dist. ☽ and ✱	63° 50' 20"
☽'s semid^{r}. red. time	15' 7"
App. dist. ☽ and ✱	63° 35' 13"

VI.

✱'s apparent alt. -	45° 9' 12"	✱'s true altitude - -	45° 8' 15"
☽'s apparent alt. -	24° 29' 44"	☽'s true altitude -	25° 17' 45"
Difference - - -	20° 39' 28"	Difference - - -	19° 50' 30"
App. dist. - - -	63° 35' 13"	½ Difference - - -	9° 55' 15"
Sum - - - - - -	84° 14' 41"		
Difference - - -	42° 55' 45"		
	42° 7' 20½"	- - ½ sum	
	21° 27' 52½"	- - ½ diff.	

Log sin - - - - - -	42° 7′ 20½″	9.8265388
Log sin - - - - - -	21° 27′ 52½″	9.5633933
Log cosine - - - -	25° 17′ 45″	9.9562230
Log cosine - - - -	45° 8′ 15″	9.8484403
C Log cos - - - - -	24° 29′ 44″	0.0409617
C Log cos - - - - -	45° 9′ 12″	0.1516804
		2|39.3872375
		19.6936187
Log sin - - - - - -	9° 55′ 15″	9.2362231
Log tan of an arc - - - - -		10.4573956
Corresponding log cos - - - -		9.5176693
Log sin - - - - - -	31° 32′ 16½″	9.7185538
	2	
True distance - - -	63° 4′ 33″	

VII.

Dist. at 9^h -	62° 49′ 15″	True dist. - -	63° 4′ 35″
Dist. at 12^h -	64° 19′ 56″	Ear. Eph. dist.	62° 49′ 15″
First diff. - -	1° 30′ 41″	Second diff. -	0° 15′ 18″

As 1° 30′ 41″ : 3^h : : 15′ 18″ : 0^h 30′ 22″

Earliest Eph. distance - - -	9^h
True time at Greenwich - -	9^h 30′ 22″
Time at the given place - -	10^h 15′ 0″
Difference of time - - - -	0^h 44′ 38″

Which converted into degrees, at the rate of 15° to an hour, gives 11° 9′ 30″, the longitude of the place east; the time at the place being greater than that at Greenwich.

Example 2.

On November 8th 1804, in longitude 24° W. by account, at 3^h 50′ 10″ P. M. the observed distance between the nearest limbs of the sun and moon was 67° 48′ 29″, the observed altitude of the moon's lower limb 31° 10′, and that of the sun's 14° 46′, the height of the eye being 12 feet; required the true longitude of the place. Ans. 24° 29′ 30″ W.

Example 3.

On October 10th 1804, in latitude 15° 15′ N. and longitude 68° E. by account, at about 6^h 4′ P. M. the watch not being well regulated, the distance of the moon's furthest limb from Fomahault was 60° 37′ 35″, the observed altitude of the moon's upper limb 46° 30′, and of the star 21° 24′, the height of the eye being 14 feet; required the true longitude of the place.

Ans. 68° 1′ 45″ E.

Example 4.

On the 13th of June 1796, in longitude 45° W. by account, the watch being well regulated, the following observations were taken:

	Times P. M.	Alt. ⊙'s lower limb.	Alt. ☽'s upper limb.	Dist. nearest limbs.
	3^h 6′ 40″	45°54′ 0″	19°32′ 0″	106°16′ 45″
	3 13 14	45 45 0	19 52 0	106 17 45
	3 17 26	45 18 45	20 5 0	106 18 30
	3 22 34	45 4 0	20 17 30	106 18 45
	3 26 46	44 48 30	20 34 0	106 19 15
	5|16 26 40	5|226 50 15	5|100 20 30	5|531 31 0
mean	3 17 20	45 22 3	20 4 6	106 18 12
Errors of the quadrants		58	1 0	2 37
True mean -		45°21′ 5″	20° 3′ 6″	106°15′ 35″

Required the true longitude, the eye being 21 feet above the sea. Ans. 42° 52′ 15″ W.

Example 5.

On the 1st of April 1796, in latitude 65° 36′ north, and longitude by account; 2°·15′ west, the watch not regulated, the following observations were taken, the eye being 18 feet above the level of the sea:

Times P. M.	Alt. ⊙'s lower limb.	Alt. ☽'s lower limb.	Dist. nearest limbs.
20ʰ 47′ 14″	22°51′	80°18′	69°36′
20 50 11	22 12	80 36	69 37
20 55 26	21 6	81 9	69 38
3|62 32 51	66 9	242 3	208 51
mean 20ʰ 50′ 57″	22° 3′	80°41′	69°37′

Required the true longitude of the place.

Ans. 2° 47′ 15″ W.

Example 6.

On the 15th of May 1796, in longitude 1° 30′ E. by account, the watch being well regulated, the following observations were taken:

Times P. M.	Alt. ☽'s lower limb.	Alt. Spica ♍.	Dist. ☽'s furth. limb.
12ʰ 36′ 14″	18° 6′ 0″	19°50′ 30″	31°30′ 45″
12 39 5	18 21 0	20 2 0	31 31 30
12 41 41	18 39 30	20 15 0	31 33 0
12 43 45	18 55 0	20 29 0	31 34 0
12 45 53	19 9 0	20 40 0	35 35 45
5|63 26 38	93 10 30	101 16 30	157 45 0
12 41 19	18 38 6	20 15 18	31 33 0
Error of quadrant	7 30+	Error of quadrant	+24
	18 45 36		31 33 24

Required the true longitude, the eye being 21 feet above the level of the sea.

Ans. Moon's true semidiameter 15′ 31″, horizontal parallax 56′ 35″, apparent alt. ☽'s centre 18° 56′ 45″, correction ☽'s altitude 50′ 46″, apparent altitude ✱'s centre 20° 10′ 56″, correction ✱'s altitude 2′ 35″, apparent distance of their centres 31° 17′ 53″, true distance 31° 11′ 43″, true longitude 15′ west.

TABLE I.

Of the Sun's Right Ascension and Declination for 1808.

Days.	January.		February.		March.	
	Right ascen.	Declin. s.	Right ascen.	Declin. s.	Right ascen.	Declin. s.
	H. M. S.	D. M. S.	H. M. S.	D. M. S.	H. M. S.	D. M. S.
1	18.43.28	23. 5. 8	20.56. 2	17.20.12	22.49. 8	7.31.28
2	18.47.53	23. 0.18	21. 0. 7	17. 3.12	22.52.52	7. 8.37
3	18.52.18	22.55. 1	21. 4.11	16.45.55	22.56.36	6.45.39
4	18.56.42	22.49.17	21. 8.14	16.28.20	23. 0.19	6.22.36
5	19. 1. 6	22.43. 6	21.12.16	16.10.28	23. 4. 2	5.59.28
6	19. 5.30	22.36.27	21.16.17	15.52.20	23. 7.44	5.36.15
7	19. 9.53	22.29.22	21.20.18	15.33.56	23.11.26	5.12.57
8	19.14.16	22.21.50	21.24.18	15.15.15	23.15. 7	4.49.37
9	19.18.38	22.13.51	21.28.16	14.56.19	23.18.49	4.26.12
10	19.22.59	22. 5.27	21.32.15	14.37. 8	23.22.29	4. 2.44
11	19.27.20	21.56.37	21.36.12	14.17.43	23.26.10	3.39.12
12	19.31.41	21.47.21	21.40. 8	13.58. 3	23.29.50	3.15.39
13	19.36. 1	21.37.40	21.44. 4	13.38.10	23.33.30	2.52. 3
14	19.40.20	21.27.33	21.47.59	13.18. 4	23.37. 9	2.28.25
15	19.44.38	21.17. 2	21.51.53	12.57.44	23.40.48	2. 4.46
16	19.48.56	21. 6. 7	21.55.47	12.37.12	23.44.27	1.41. 6
17	19.53.13	20.54.47	21.59.40	12.16.28	23.48. 6	1.17.25
18	19.57.30	20.43. 3	22. 3.32	11.55.31	23.51.45	0.53.43
19	20. 1.46	20.30.55	22. 7.23	11.34.24	23.55.23	0.30. 1
20	20. 6. 1	20.18.25	22.11.14	11.13. 5	23.59. 2	0. 6.20
						North.
21	20.10.15	20. 5.31	22.15. 4	10.51.36	0. 2.40	0.17.21
22	20.14.29	19.52.15	22.18.54	10.29.57	0. 6.18	0.41. 2
23	20.18.42	19.38.36	22.22.42	10. 8. 8	0. 9.56	1. 4.40
24	20.22.54	19.24.36	22.26.31	9.46.10	0.13.34	1.28.17
25	20.27. 5	19.10.14	22.30.18	9.24. 2	0.17.12	1.51.52
26	20.31.16	18.55.31	22.34. 5	9. 1.47	0.20.50	2.15.25
27	20.35.26	18.40.27	22.37.52	8.39.23	0.24.28	2.38.54
28	20.39.34	18.25. 3	22.41.38	8.16.52	0.28. 6	3. 2.20
29	20.43.43	18. 9.19	22.45.33	7.54.13	0.31.44	3.25.43
30	20.47.50	17.53.15	—	—	0.35.22	3.49. 2
31	20.51.56	17.36.53	—	—	0.39. 1	4.12.17

TABLE I.

Of the Sun's Right Ascension and Declination for 1808.

Days.	April.		May.		June.	
	Right ascen.	Declin. N.	Right ascen.	Declin. N.	Right ascen.	Declin. N.
	H.M.S.	D.M.S.	H.M.S.	D.M.S.	H.M.S.	D.M.S.
1	0.42.39	4.35.26	2.33.52	15. 6.29	4.36.34	22. 4.33
2	0.46.17	4.58.31	2.37.41	15.24.29	4.40.39	22.12.30
3	0.49.55	5.21.30	2.41.31	15.42.14	4.44.45	22.20. 4
4	0.53.34	5.44.24	2.45.21	15.59.43	4.48.52	22.27.14
5	0.57.13	6. 7.12	2.49.11	16.16.56	4.52.58	22.34. 1
6	1. 0.51	6.29.53	2.53. 3	16.33.53	4.57. 5	22.40.24
7	1. 4.30	6.52.27	2.56.55	16.50.33	5. 1.13	22.46.23
8	1. 8.10	7.14.54	3. 0.47	17. 6.56	5. 5.20	22.51.59
9	1.11.49	7.37.14	3. 4.40	17.23. 3	5. 9.28	22.57.10
10	1.15.29	7.59.26	3. 8.33	17.38.52	5.13.36	23. 1.58
11	1.19. 9	8.21.30	3.12.28	17.54.23	5.17.45	23. 6.21
12	1.22.49	8.43.26	3.16.22	18. 9.37	5.21.53	23.10.19
13	1.26.30	9. 5.13	3.20.18	18.24.32	5.26. 2	23.13.54
14	1.30.11	9.26.51	3.24.13	18.39. 9	5.30.11	23.17. 3
15	1.33.52	9.48.20	3.28.10	18.53.27	5.34.20	23.19.49
16	1.37.34	10. 9.39	3.32. 7	19. 7.26	5.38.29	23.22. 9
17	1.41.16	10.30.49	3.36. 5	19.21. 6	5.42.39	23.24. 5
18	1.44.58	10.51.48	3.40. 3	19.34.26	5.46.48	23.25.36
19	1.48.41	11.12.36	3.44. 2	19.47.27	5.50.58	23.26.42
20	1.52.24	11.33.14	3.48. 1	20. 0. 7	5.55. 8	23.27.24
21	1.56. 8	11.53.41	3.52. 1	20.12.27	5.59.17	23.27.41
22	1.59.52	12.13.56	3.56. 1	20.24.27	6. 3.27	23.27.32
23	2. 3.37	12.33.58	4. 0. 2	20.36. 5	6. 7.36	23.26.59
24	2. 7.22	12.53.49	4. 4. 4	20.47.23	6.11.46	23.26. 2
25	2.11. 8	13.13.27	4. 8. 6	20.58.19	6.15.55	23.24.39
26	2.14.54	13.32.52	4.12. 9	21. 8.53	6.20. 5	23.22.52
27	2.18.40	13.52. 3	4.16.12	21.19. 6	6.24.14	23.20.40
28	2.22.27	14.11. 1	4.20.15	21.28.56	6.28.23	23.18. 3
29	2.26.15	14.29.46	4.24.19	21.38.24	6.32.32	23.15. 2
30	2.30. 3	14.48.15	4.28.24	21.47.30	6.36.40	23.11.37
31	—	—	4.32.29	21.56.13	—	—

TABLE I.

Of the Sun's Right Ascension and Declination for 1808.

Days.	July.		August.		September.	
	Right ascen.	Declin. N.	Right ascen.	Declin. N.	Right ascen.	Declin. N.
	H. M. S.	D. M. S.	H. M. S.	D. M. S.	H. M. S.	D. M. S.
1	6.40.48	23. 7.47	8.45.37	18. 1.56	10.41.41	8.16.33
2	6.44.56	23. 3.38	8.49.29	17.46.40	10.45.18	7.54.41
3	6.49. 4	22.58.54	8.53.22	17.31. 6	10.48.56	7.32.41
4	6.53.12	22.53.52	8.57.13	17.15.16	10.52.33	7.10.34
5	6.57.19	22.48.26	9. 1. 4	16.59. 9	10.56. 9	6.48.20
6	7. 1.25	22.42.36	9. 4.54	16.42.45	10.59.46	6.26. 0
7	7. 5.32	22.36.23	9. 8.44	16.26. 5	11. 3.22	6. 3.33
8	7. 9.38	22.29.46	9.12.33	16. 9.10	11. 6.59	5.41. 0
9	7.13.43	22.22.45	9.16.22	15.51.58	11.10.35	5.18.21
10	7.17.49	22.15.22	9.20.10	15.34.32	11.14.11	4.55.36
11	7.21.53	22. 7.36	9.23.57	15.16.50	11.17.46	4.32.47
12	7.25.58	21.59.26	9.27.44	14.58.53	11.21.22	4. 9.52
13	7.30. 2	21.50.54	9.31.30	14.40.42	11.24.58	3.46.53
14	7.34. 5	21.42. 0	9.35.16	14.22.17	11.28.33	3.23.50
15	7.38. 8	21.32.43	9.39. 1	14. 3.38	11.32. 9	3. 0.43
16	7.42.11	21.23. 4	9.42.46	13.44.45	11.35.44	2.37.33
17	7.46.13	21.13. 4	9.46.30	13.25.40	11.39.20	2.14.19
18	7.50.14	21. 2.41	9.50.14	13. 6.21	11.42.55	1.51. 2
19	7.54.15	20.51.58	9.53.58	12.46.50	11.46.31	1.27.43
20	7.58.16	20.40.53	9.57.40	12.27. 6	11.50. 7	1. 4.22
21	8. 2.16	20.29.27	10. 1.23	12. 7.11	11.53.42	0.40.59
22	8. 6.15	20.17.40	10. 5. 5	11.47. 4	11.57.18	0.17.35
						South.
23	8.10.14	20. 5.33	10. 8.46	11.26.46	12. 0.54	0. 5.51
24	8.14.12	19.53. 6	10.12.27	11. 6.17	12. 4.30	0.29.17
25	8.18.10	19.40.19	10.16. 8	10.45.37	12. 8. 6	0.52.43
26	8.22. 7	19.27.12	10.19.48	10.24.47	12.11.42	1.16. 9
27	8.26. 4	19.13.46	10.23.28	10. 3.47	12.15.19	1.39.34
28	8.29.59	19. 0. 1	10.27. 7	9.42.38	12.18.55	2. 2.59
29	8.33.55	18.45.57	10.30.46	9.21.20	12.22.32	2.26.23
30	8.37.49	18.31.36	10.34.24	8.59.52	12.26. 9	2.49.45
31	8.41.43	18.16.54	10.38. 3	8.38.17		

TABLE I.

Of the Sun's Right Ascension and Declination for 1808.

Days.	October.		November.		December.	
	Right ascen.	Declin. s.	Right ascen.	Declin. s.	Right ascen.	Declin. s.
	H. M. S.	D. M. S.	H. M. S.	D. M. S.	H. M. S.	D. M. S.
1	12.29.47	3.13. 6	14.26. 5	14.28.54	16.29.58	21.50.53
2	12.33.24	3.36.24	14.30. 1	14.48. 1	16.34.18	21.59.57
3	12.37. 2	3.59.39	14.33.57	15. 6.54	16.38.38	22. 8.36
4	12.40.40	4.22.52	14.37.54	15.25.32	16.42.59	22.16.49
5	12.44.19	4.46. 2	14.41.53	15.43.54	16.47.20	22.24.36
6	12.47.58	5. 9. 8	14.45.52	16. 2. 1	16.51.42	22.31.56
7	12.51.37	5.32.10	14.49.51	16.19.52	16.56. 5	22.38.51
8	12.55.17	5.55. 9	14.53.52	16.37.26	17. 0.28	22.45.18
9	12.58.57	6.18. 2	14.57.54	16.54.43	17. 4.51	22.51.19
10	13. 2.38	6.40.51	15. 1.56	17.11.44	17. 9.15	22.56.53
11	13. 6.19	7. 3.35	15. 5.59	17.28.27	17.13.40	23. 2. 0
12	13.10. 1	7.26.12	15.10. 4	17.44.51	17.18. 4	23. 6.39
13	13.13.43	7.48.44	15.14. 9	18. 0.58	17.22.30	23.10.51
14	13.17.26	8.11.10	15.18.14	18.16.45	17.26.55	23.14.35
15	13.21. 9	8.33.29	15.22.21	18.32.13	17.31.21	23.17.51
16	13.24.53	8.55.40	15.26.29	18.47.22	17.35.47	23.20.39
17	13.28.38	9.17.44	15.30.37	19. 2.11	17.40.13	23.22.59
18	13.32.28	9.39.40	15.34.47	19.16.39	17.44.39	23.24.51
19	13.36. 8	10. 1.28	15.38.57	19.30.46	17.49. 6	23.26.14
20	13.39.55	10.23. 7	15.43. 8	19.44.32	17.53.33	23.27.10
21	13.43.42	10.44.36	15.47.20	19.57.56	17.57.59	23.27.36
22	13.47.29	11. 5.56	15.51.32	20.10.59	18. 2.26	23.27.35
23	13.51.18	11.27. 6	15.55.46	20.23.39	18. 6.53	23.27. 5
24	13.55. 7	11.48. 6	16. 0. 0	20.35.56	18.11.20	23.26. 7
25	13.58.56	12. 8.54	16. 4.14	20.47.50	18.15.46	23.24.41
26	14. 2.47	12.29.31	16. 8.30	20.59.21	18.20.13	23.22.46
27	14. 6.38	12.49.57	16.12.46	21.10.28	18.24.39	23.20.24
28	14.10.30	13.10.10	16.17. 3	21.21.11	18.29. 5	23.17.33
29	14.14.23	13.30.11	16.21.21	21.31.30	18.33.31	23.14.14
30	14.18.16	13.49.59	16.25.39	21.41.24	18.37.57	23.10.27
31	14.22.10	14. 9.34			18.42.22	23. 6.12

TABLE II.

Refraction in Altitude and Dip of the Horizon.

Apparent altitude.	Refraction.	Apparent altitude.	Refraction.	Apparent altitude.	Refraction	Apparent altitude.	Refraction.	App. alt.	Refraction.	App. alt.	Refraction.	Height of eye.	Dip of the hor. at sea.
D. M.	M. S.	D. M.	M. S.	D. M.	M. S.	D. M.	M. S.	D.	M. S.	D.	M. S.		
0. 0	33. 0	2.30	16.24	6.30	7.51	12.20	4.16	30	1.38	60	0.33	Ft.	M. s.
0. 5	32.10	2.35	16. 4	6.40	7.40	12.40	4. 9	31	1.35	61	0.32	1	0.57
0.10	31.22	2.40	15.45	6.50	7.30	13. 0	4. 3	32	1.31	62	0.30	2	0.21
0.15	30.35	2.45	15.27	7. 0	7.20	13.20	3.57	33	1.28	63	0.29	3	0.39
0.20	29.50	2.50	15. 9	7.10	7.11	13.40	3.51	34	1.24	64	0.28	4	1.55
0.25	29. 6	2.55	14.52	7.20	7. 2	14. 0	3.45	35	1.21	65	0.26	5	2. 8
0.30	28.22	3. 0	14.36	7.30	6.53	14.20	3.40	36	1.18	66	0.25	6	2.20
0.35	27.41	3. 5	14.20	7.40	6.45	14.40	3.35	37	1.16	67	0.24	7	2.31
0.40	27. 0	3.10	14. 4	7.50	6.37	15. 0	3.30	38	1.13	68	0.23	8	2.42
0.45	26.20	3.15	13.49	8. 0	6.29	15.30	3.24	39	1.10	69	0.22	9	2.52
0.50	25.42	3.20	13.34	8.10	6.22	16. 0	3.17	40	1. 8	70	0.21	10	3. 1
0.55	25. 5	3.25	13.20	8.20	6.15	16.30	3.10	41	1. 5	71	0.19	11	3.10
1. 0	24.29	3.30	13. 6	8.30	6. 8	17. 0	3. 4	42	1. 3	72	0.18	12	3.18
1. 5	23.54	3.40	12.40	8.40	6. 1	17.30	2.59	43	1. 1	73	0.17	13	3.26
1.10	23.20	3.50	12.15	8.50	5.55	18. 0	2.54	44	0.59	74	0.16	14	3.34
1.15	22.47	4. 0	11.51	9. 0	5.48	18.30	2.49	45	0.57	75	0.15	15	3.42
1.20	22.15	4.10	11.29	9.10	5.42	19. 0	2.44	46	0.55	76	0.14	16	3.49
1.25	21.44	4.20	11. 8	9.20	5.36	19.30	2.39	47	0.53	77	0.13	17	3.56
1.30	21.15	4.30	10.48	9.30	5.31	20. 0	2.35	48	0.51	78	0.12	18	4. 3
1.35	20.46	4.40	10.29	9.40	5.25	20.30	2.31	49	0.49	79	0.11	19	4 10
1.40	20.18	4.50	10.11	9.50	5.20	21. 0	2.27	50	0.48	80	0.10	20	4.16
1.45	19.51	5. 0	9.54	10. 0	5.15	21.30	2.24	51	0.46	81	0. 9	21	4.22
1.50	19.25	5.10	9.38	10.15	5. 7	22. 0	2.20	52	0.44	82	0. 8	22	4.28
1.55	19. 0	5.20	9.23	10.30	5. 0	23. 0	2.14	53	0.43	83	0. 7	23	4.34
2. 0	18.35	5.30	9. 8	10.45	4.53	24. 0	2. 7	54	0.41	84	0. 6	24	4.40
2. 5	18.11	5.40	8.54	11. 0	4.47	25. 0	2. 2	55	0.40	85	0. 5	26	4.52
2.10	17.48	5.50	8.41	11.15	4.40	26. 0	1.56	56	0.38	86	0. 4	28	5. 3
2.15	17.26	6. 0	8.28	11.30	4.34	27. 0	1.51	57	0.37	87	0. 3	30	5.14
2.20	17. 4	6.10	8.15	11.45	4.29	28. 0	1.47	58	0.35	88	0. 2	35	5.39
2.25	16.44	6.20	8. 3	12. 0	4.23	29. 0	1.42	59	0.34	89	0. 1	40	6. 2

TABLE III.

Right Ascension and Declination of 36 principal Stars.

1806.	Mag.	Mean R. A. in Sidereal Time.	Ann. variation.	Mean Declin.	Annual variation.
		h. m. s.	s. +	° ′ ″	s.
γ Pegasi......	2	0 3 15,40	3,069	14 6 24,90 N.	+20,20
α Arietis	2.3	1 56 15,66	3,347	22 32 24,98 N.	+17,47
α Ceti........	2	2 52 8,88	3,115	3 19 22,40 N.	+14,75
Aldebaran ..	1	4 24 48,00	3,426	16 6 31,40 N.	+ 8,00
Capella	1	5 2 22,62	4,415	45 47 5,88 N.	+ 4,57
Rigel.......	1	5 55 13,11	2,876	8 25 59,32 S.	− 4,92
β Tauri	2	5 14 2,17	3,781	28 25 52,56 N.	+ 3,91
α Orionis	1	5 44 40,23	3,243	7 21 38,16 N.	+ 1,49
Sirius	1	6 36 36,06	2,653	16 27 21,54 S.	+ 4,21
Castor......	2	7 22 11,92	3,853	32 18 3,76 N.	− 7,06
Procyon ..	1.2	7 29 8,17	3,142	5 42 51,48 N.	− 8,53
Pollux......	2	7 33 25,43	3,688	28 29 2,58 N.	− 7,93
α Hydræ......	2	9 18 3,08	2,946	7 49 19,70 S.	+15,10
Regulus	1	9 58 1,65	3,212	12 54 41,74 N.	−17,19
β Leonis	1.2	11 39 9,14	3,067	15 39 25,24 N.	−20,04
β Virginis	3	11 40 35,27	3,125	2 51 32,42 N.	−20,22
α Virginis	1	13 14 59,29	3,147	10 8 29,80 S.	+18,80
Arcturus	1	14 6 48,83	2,728	20 11 59,41 N.	−18,79
^{1}α Libræ	2	14 39 58,66	3,296	15 10 42,66 S.	+15,19
^{2}α Libræ	2	14 40 9,99	3,297	15 12 26,84 S.	+15,21
α Coronæ....	2.3	15 26 28,63	2,545	27 22 34,54 N.	−12,49
α Serpentis....	2	15 34 43,17	2,945	7 2 48,60 N.	−11,70
Antares	1	16 17 32,06	3,658	25 59 4,92 S.	+ 8,48
α Hercules ..	2.3	17 5 48,33	2,731	14 37 26,48 N.	− 4,48
α Ophiuchi....	2	17 25 55,91	2,776	12 42 47,88 N.	− 3,03
α Lyræ :.....	1	18 30 22,08	2,027	38 36 36,34 N.	+ 2,91
α Aquilæ......	3	19 37 1,91	2,846	10 9 6,72 N.	+ 8,38
α Aquilæ	1.2	19 41 18,83	2,925	8 22 2,64 N.	+ 9,11
β Aquilæ	3.4	19 45 46,85	2,944	5 56 1,28 N.	+ 8,57
^{1}α Capricorni ..	4	20 6 53,03	3,336	13 5 39,70 S.	−10,80
^{2}α Capricorni ..	3	20 7 16,83	3,339	13 7 58,36 S.	−10,81
α Cygni	1.2	20 34 49,06	2,038	44 35 33,84 N.	+12,56
α Aquarii	3	21 55 48,75	3,081	1 15 15,66 S.	−17,36
Fomalhaut	1.2	22 46 54,18	3,343	30 38 26,30 S.	−19,10
α Pegasi......	2	22 55 6,12	2,973	14 9 59,32 N.	+19,43
α Andromedæ..	2	23 58 22,89	3,070	27 58 34,24 N.	+19,99

TABLE IV.

Augmentation of the Moon's Semidiameter.

☽'s alt.	Augm.	☽'s alt.	Augm.
D.	S.	D.	S.
0	0	40	10
5	1	45	11
10	3	50	12
15	4	55	13
20	6	60	14
25	7	70	15
30	8	80	16
35	9		

TABLE V.

The Sun's Parallax in Altitude.

⊙'s alt.	Parallax.	⊙'s alt.	Parallax.
D.	S.	D.	S.
0	9	60	4
10	9	65	4
20	8	70	3
30	8	75	2
40	7	80	2
50	6	85	1
55	5	90	0

TABLE VI.

To convert Time into Longitude.

Hours.	Degrees.	M.	D.	M.	M.	D.	M.
		S.	M.	S.	S.	M.	S.
		T.	S.	T.	T.	S.	T.
1	15	1	0	15	31	7	45
2	30	2	0	30	32	8	0
3	45	3	0	45	33	8	15
4	60	4	1	0	34	8	30
5	75	5	1	15	35	8	45
6	90	6	1	30	36	9	0
7	105	7	1	45	37	9	15
8	120	8	2	0	38	9	30
9	135	9	2	15	39	9	45
10	150	10	2	30	40	10	0
11	165	11	2	45	41	10	15
12	180	12	3	0	42	10	30
13	195	13	3	15	43	10	45
14	210	14	3	30	44	11	0
15	225	15	3	45	45	11	15
16	240	16	4	0	46	11	30
17	255	17	4	15	47	11	45
18	270	18	4	30	48	12	0
19	285	19	4	45	49	12	15
20	300	20	5	0	50	12	30
21	315	21	5	15	51	12	45
22	330	22	5	30	52	13	0
23	345	23	5	45	53	13	15
24	360	24	6	0	54	13	30
		25	6	15	55	13	45
		26	6	30	56	14	0
		27	6	45	57	14	15
		28	7	0	58	14	30
		29	7	15	59	14	45
		30	7	30	60	15	0

TABLE VII.

To convert Longitude into Time.

D.	H.	M.	D.	H.	M.	Degrees.	Hours.	Minutes.
M.	M.	S.	M.	M.	S.			
S.	S.	T.	S.	S.	T.			
1	0	4	31	2	4	70	4	40
2	0	8	32	2	8	80	5	20
3	0	12	33	2	12	90	6	0
4	0	16	34	2	16	100	6	40
5	0	20	35	2	20	110	7	20
6	0	24	36	2	24	120	8	0
7	0	28	37	2	28	130	8	40
8	0	32	38	2	32	140	9	20
9	0	36	39	2	36	150	10	0
10	0	40	40	2	40	160	10	40
11	0	44	41	2	44	170	11	20
12	0	48	42	2	48	180	12	0
13	0	52	43	2	52	190	12	40
14	0	56	44	2	56	200	13	20
15	1	0	45	3	0	210	14	0
16	1	4	46	3	4	220	14	40
17	1	8	47	3	8	230	15	20
18	1	12	48	3	12	240	16	0
19	1	16	49	3	16	250	16	40
20	1	20	50	3	20	260	17	20
21	1	24	51	3	24	270	18	0
22	1	28	52	3	28	280	18	40
23	1	32	53	3	32	290	19	20
24	1	36	54	3	36	300	20	0
25	1	40	55	3	40	310	20	40
26	1	44	56	3	44	320	21	20
27	1	48	57	3	48	330	22	0
28	1	52	58	3	52	340	22	40
29	1	56	59	3	56	350	23	20
30	2	0	60	4	0	360	24	0

MISCELLANEOUS ASTRONOMICAL PROBLEMS.

1. In what latitude north, will the shortest day be just $\frac{3}{5}$ of the longest? Ans. Lat. 41° 23′ 7″.

2. At what time of the day, in the month of May, when the sun's declination is 20° 16′ 32″, will the shadow of a perpendicular object be just equal to its length? Ans. $9^{h}\ 13'\ 16''$ A. M.

3. In what latitude, on the first of June, will the sun's altitude, when due east, be double his altitude at 6 o'clock? Ans. Lat. 49° 6′ S.

4. At what time in the afternoon, on the 9th of June, is the sun's altitude exactly the same in the latitudes of 50° and 60° north? Ans. $4^{h}\ 49'\ 11''$ P. M.

5. On the 24th of May, in latitude 50° 12′ N. required the time it will take for the body of the sun to rise out of the horizon. Ans. 3′ 58″.

6. On July 1st 1792, in latitude 57° 9′ N. and longitude 2° 8′ W. the stars Vega and Altair were observed on the same vertical circle, at $10^{h}\ 9'$ per watch; required the apparent time of observation, and the error of the watch.

Ans. Apparent time $10^{h}\ 8'\ 14''$, watch too fast 46″.

7. On October 25th 1792, in longitude 21° E. by account, the interval in mean time, between the rising of Aldebaran and Rigel, was $3^{h}\ 17'\ 30''$; required the apparent time of the rising of Aldebaran.

Ans. $6^{h}\ 36'\ 4''$.

8. On July 4th 1804, in latitude 35° 48′ S. and longitude 23° 26′ E. the mean times per watch, of

several equal altitudes of Altair, were $8^h\ 21'\ 15\frac{1}{2}''$ and $14^h\ 31'\ 41\frac{1}{2}''$; required the error of the watch for apparent time. Ans. Watch too fast 53″.

9. On the 19th of December 1804, in latitude 37° 46′ N. longitude 21° 15′ E. a certain phænomenon was observed at the same instant that the altitude of Arcturus, when east of the meridian, was found to be $34°\ 6\frac{2}{5}'$; required the apparent time of observation, the height of the eye being 10 feet.

Ans. Apparent time $16^h\ 33'\ 34''$.

10. On February 14th 1792, in latitude 43° 26′ N. and longitude 54° 16′ W. the altitude of Jupiter, at $12^h\ 23'\ 5''$ per watch, was found to be 16° 9.7′; required the error of the watch, the height of the eye being 14 feet. Ans. Watch exact.

11. In latitude 51° 31′ N. having found the azimuth of an object to be S. 48° 10′ E.; it is required to find the error in the apparent time of observation, corresponding to a supposed error of 10′ in the altitude.

Ans. 1′ 26″.

12. In latitude 54° 42′ by account, having found the azimuth of an object to be 26° 17′; it is required to find the error in the apparent time, answering to a supposed error of 10′ in the latitude. Ans. 2′ 20″.

13. At $9^h\ 51'\ 58''$ A. M. per watch, the correct altitude of the sun's centre was 21° 11′, at $10^h\ 48'\ 54''$ it was 24° 40′, and at $11^h\ 29'\ 42''$ it was 26°; required the apparent time when the greatest altitude was observed, and the error of the watch.

Ans. Apparent time $11^h\ 29'\ 16''$; watch too fast 26″.

14. On November 21st 1792, at 10 hours, reduced time, the true distance between α Orionis and the moon's centre was found to be 105° 26′ 14″; from which it is required to find the moon's longitude.

Ans. 11ˢ 10° 5′ 50″.

15. On December 30th 1792, the moon's eastern limb was observed to pass the meridian at 13ʰ 53′ 33.8″; from which it is required to find the longitude of the place of observation. Ans. 78° 15′ E.

16. On Nov. 13th 1804, in latitude 45° 35′ 25″ N. and longitude by account 20° W. the meridian altitude of the moon's lower limb was found to be 48° 33′ 40″; required the longitude of the place of observation, the height of the eye being 15 feet. Ans. 19° 45′ W.

17. On October 2d 1800, the beginning of a lunar eclipse was observed at 6ʰ 1½′ P. M. per watch, and the end at 12ʰ 52′ P. M.; required the longitude of the place of observation, the watch, for apparent time, being 13¼′ too slow. Ans. 41° 30′ W.

18. Required the altitude and longitude of the nonagesimal degree, in latitude 57° 8.9′ N. and longitude in time 8′ 40″ W. on November 26th 1787, at 11ʰ 18′ 8″ apparent time.

Ans. Altitude 53° 22′, longitude 65° 24¼′.

19. On June 3d 1788, in latitude 57° 9′ N. and longitude, by account, 8′ 32″ in time, the beginning of a solar eclipse was observed at 19ʰ 33′ 19″ apparent time, and the end at 20ʰ 49′ 29″; required the longitude of the place of observation. Ans. 2° 9′ W.

20. On November 26th 1787, in latitude 57° 9′ N.

longitude by account 8′ 32″ w. in time, the immersion of η ♊, in an occultation of that star by the moon, was observed at 11° 18′ 8″ apparent time, and the emersion at 12° 23′ 12″; required the longitude of the place of observation.

Ans. Long. in time 7′ 25″ w.

21. On December 9th 1809, an emersion of the first satellite of Jupiter was observed at 16ʰ 58′ 35″ per watch, which was 3′ 58″ too slow for apparent time; required the longitude of the place of observation. Ans. 12° 35′ w.

22. On November 6th 1804, in longitude 158° w. the meridian altitude of the sun's lower limb was 87° 37′ n.; required the latitude of the place, the height of the eye being 12 feet. Ans. 18° 19½′ s.

23. On December 25th 1804, the meridian altitude of Saturn was found to be 68° 42′ n.; required the latitude of the place, the height of the eye being 15 feet. Ans. 25° 11′ s.

24. On December 14th 1804, in longitude 30° w. the meridian altitude of the moon's lower limb was found to be 81° 15′ n.; required the latitude of the place, the height of the eye being 16 feet.

Ans. 15° 17′ n.

25. In north latitude, at 11ʰ 10′ and at 12ʰ 40′ per watch, the altitude of the sun's lower limb, when corrected, was 26° 55′, and his declination 5° 17′ s.; required the latitude of the place. Ans. 57° 9′ n.

26. At 9ʰ 23′ 20″ a. m. apparent time, the true alti-

tude of the sun's centre was 34° 29′, and at 11^{h} 9′ 32″ it was 42° 19′; required the latitude and declination.
Ans. Lat. 57° 7′ N. Dec. 10° 27′ N.

27. On June 4th 1804, in latitude by account 37′ N. at 10^{h} 29′ in the forenoon, per watch, the correct altitude of the sun's centre was 65° 24′, and at 12^{h} 31′ it was 74° 8′; required the true latitude.
Ans. 36° 57′ N.

28. On January 1st 1805, in north latitude, the true altitude of Capella was 69° 23′, and, at the same instant, the true altitude of Sirius was 16° 19′; required the true latitude. Ans. 57° 8′ N.

29. On the 12th of December 1804, being in north latitude, and in longitude 24° W. by account, at 5^{h} 24′ P. M. per watch, the altitude of the moon's lower limb was 41° 33′, and at 7^{h} 12′ it was 52° 56′; required the true latitude of the place, the height of the eye being 20 feet. Ans. 48° 52′.

30. On the 15th of May 1804, in latitude 33° 10′ N. and longitude 18° W. about 5 o'clock A. M. the sun was observed to rise E. by N.; required the variation of the compass. Ans. 11° 26′ W.

31. On November 19th 1806, in latitude 50° 22′ N. and longitude 24° 30′ W. about three quarters past 8 o'clock A. M. the altitude of the sun's lower limb was 8° 10′, and his bearing per compass S. 21° 18′ E.; required the variation, the height of the eye being 20 feet. Ans. 24° 12′ W.

32. On August 19th 1810, in latitude 41° 46′ N.

longitude 144° 20′ W. at 9^h 13′ 44″ A. M. apparent time, the magnetic azimuth of the sun's centre was S. 80° 20′ W. required the variation. Ans. 16° 42′ E.

33. On August 26th 1809, in the forenoon, the sun's magnetic azimuth was S. 22° 41′ E. and his correct central altitude 33° 14′; and some time afterwards, the magnetic azimuth was S. 14° 53′ W. and the true altitude 42° 36′; required the latitude and variation.

Ans. Lat. 57° 8½′ N. variation 29° 31′ W.

34. Required the latitude and longitude of a star, its right ascension being 16^h 14′, its declination 25° 51′ N., and the obliquity of the ecliptic 23° 28′.

Ans. Lat. 46° 6½′ N. Longitude 234° 36′.

35. In latitude 20° N. the gnomon of an horizontal dial being perpendicular to the plane of the horizon, it is required to find at what hour in the afternoon, on the longest day, the shadow of the gnomon will stand still, and how many degrees it will run back.

Ans. Stands still at 2^h 12′ 8″, and runs back 12° 32′.

36. At London, on the 10th of December 1780, at what time of the night were the stars Aldebaran and Rigel on the same azimuth circle?

Ans. 9^h 32′ 23″ evening.

37. At what time in the evening were the stars Betelguese and Pollux on the same almacanter, or of one common altitude above the horizon of London, on the 10th of December 1780?

Ans. 9^h 12′ 24″.

38. Being at sea, in an unknown latitude, I ob-

served the star Schedar in Cassiopeia, and Almaach in Andromeda, to have the same azimuth, when the altitude of Schedar was 37° 15′; required the latitude of the place. Ans. 50° 44′.

39. Being at sea, in an unknown place, the star Aldebaran was observed to rise 3^{h} 15′ later than the bright star in Aries; required the latitude of the place. Ans. 54° 38′.

40. Some time in the month of May 1780, at a place in the Western ocean, the sun's meridian altitude was observed to be 62°, and 1^{h} 48′ 14″ after it was found to be 54° 30′; required the day of the month and the latitude of the place.

Ans. 19th May, latitude 48° 0′ 12″ N.

41. Some time in July, in N. latitude, the sun was observed to rise at 4^{h} 24′ 36″ A. M. and his altitude, at noon, in the same place, was 62°; required the day of the month and the latitude of the place.

Ans. July 23d, latitude 48° 0′ 10″ N.

42. At a place in the Western ocean, in the month of May, the sun's altitude, at 6^{h} A. M. was 14° 43½′, and at 8^{h} 8′ 11″ it was 36°; required the latitude of the place, and the day of the month.

Ans. Latitude 48° N. May 19th.

43. At a place in the Western ocean, the sun was observed, at rising, to be 59° 15′ 40″ from the true north point of the horizon, and his altitude at 6^{h} A. M. was 14° 43½′; required the latitude and declination.

Ans. Latitude 48° N. Declination 20°.

44. At a place in the Western ocean, some time in May 1763, the altitude of the sun's centre, at 8ʰ 8′ 11″ A. M. was 36°, and at 9ʰ 10′ 51″ it was 46°; required the latitude and declination.

Ans. Latitude 48° N. Declination 19° 59′.

45. Some time in July 1763, three descending altitudes of the sun were found to be 54½°, 46°, and 36°, and the intervals of time between them 60′ 55″ and 62′ 40″; required the times when the observations were made. Ans. Time 2d obser. 2ʰ 48′ 59″,
Time 3d obser. 3ʰ 51′ 39″.

46. It is required to find the declination of a plane, upon which the sun, on June 10th, in latitude 51° 32′ N. will continue 9ʰ 50′. Ans. 60° 31′.

47. To determine the latitude of the place, where the shadow of the gnomon, on an horizontal dial, will move, between 3 and 4 o'clock, with the greatest velocity possible. Ans. Latitude 49° 29′ 43″.

48. It is required to find the greatest error of an horizontal dial, made for a place in latitude 51° 32′ N. but placed in latitude 54°. Ans. 4′ 27″.

49. It is required to find, in latitude 66° N. when the continuance of the morning and evening twilight is just equal to the length of the day.

Ans. January 28th and November 12th.

50. In what latitude, on the 21st of June, will the sun be due east, when he has run half his course between the time of his rising and noon?

Ans. Latitude 64° 35′ 48″ N.

51. At what time, on the 10th of June, in latitude 12° 30′ N. will the sun's azimuth be the greatest possible? Ans. 8h 5′ A. M. or 3h 55′ P. M.

52. In what points of the ecliptic, between ♈ and ♋, does the sun's longitude exceed his right ascension the greatest possible? Ans. In 16° 14′ 16″ of Taurus.

53. On what day of the year, in latitude 51° 32′ N. does the sun's azimuth increase the most possible in 2 hours after rising? Ans. Nov. 19th or Jan. 22d.

54. On what day of the year, in latitude 51° 32′ N. does the length of the afternoon exceed that of the forenoon the most possible, reckoning the day to begin at sun-rise and to end at sun-set?

Ans. 19th April, when sun's long. is 29° 32′.

55. A person has an horizontal dial, made for the latitude of 51° 32′ N. how must he set it so that it shall go true in latitude 53° 15′ N?

Ans. Directly N. and S. inclining 88° 17′ from the zenith.

56. To determine a place on the earth, where a degree of the meridian is equal to a degree of the equator; supposing the ratio of the two axes to be as 229 to 230. Ans. From lat. 54° 17′ 38″ to lat. 55° 17′ 38″.

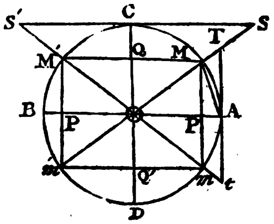
57. The latitude of London being 31° 30′, the latitude and longitude of Moscow 55° 45′ and 38° 0′, and those of Constantinople 41° 30′ and 29° 15′ respectively, it is required to find the latitude and longitude of a place that shall be equidistant from all the three.

Ans. Lat. of the place 51° 17′ and long. 19° 13″.

OF THE SIGNS OF TRIGONOMETRICAL QUANTITIES.

As no account has been given, in the former part of this work, of the change of signs which the various trigonometrical lines are liable to undergo, according to the magnitude of the arc to which they belong, it will be here proper to enter into some explanation of this part of the subject, which requires to be particularly attended to in many analytical investigations.

1. For this purpose, let A C B D be a circle, having the sines, tangents, &c. represented as in the figure; and suppose one of the extremities A, of an arc A M to remain fixed, while the other extremity M passes successively over the circumference of the circle, from A through C, B, D to A again.



Then as the sine M P continually recedes from A, till the point M arrives at B, and afterwards approaches towards A, on the other side of the diameter A B, till it is united with it again, it is plain that the sines of all arcs, in the first semicircle A C B, are affirmative or +, and that the sines of all arcs, in the second semicircle B D A, are negative, or —.

It is also evident, that the sine M P increases from 0, during the 1st quadrant A C, till, at the end of it, it becomes equal to the radius; and that it decreases during the 2d quadrant C B, till it again becomes O. After this, it passes to the other side of the diameter, and increases negatively during the 3d quadrant B D, till it becomes equal to − radius; and then decreases negatively during the 4th quadrant D A, till it becomes O, as before.

It appears, likewise, from an inspection of the figure, that the sine M P increases faster in the first part A M, of the quadrant A C, than when it approaches near the end of it at C; and, on the contrary, that it decreases more slowly in the first part C M′ of the 2d quadrant C B, than when it arrives near the end of it at B; owing to the convexity of the circle being more or less favourable to this variation.

2. In like manner, the cosine O P, being referred to the centre O, will become negative as often as it passes that point; and as this takes place both when the arc A M becomes greater than A C, and when, by its further increase, it is greater than A C B D, it is evident that the cosines of all arcs in the 1st and 4th quadrants A C, D A will be positive, or +, and that the cosines of all arcs in the 2d and 3d quadrants C B, B D will be negative, or −.

It is also plain, that the cosine O P is equal to radius when the arc A M is O, and that it continually decreases during the 1st quadrant A C, till, at the end of it, it is O. After this, it increases negatively during the 2d quadrant C B, at the end of which it is equal to − radius. It then decreases negatively during the 3d quadrant B D, at the end of which it is o; and in the 4th quadrant D A, it again becomes positive, and increases till it is equal to radius, as before.

And since the cosine of the arc A M is equal to the sine of its complement M C, it follows, reversedly, from what has been said respecting the variation of the sines, that the cosine O P increases slower in the first part A M of the quadrant A C, than when it approaches near the end of it, at C; and, on the contrary, that it decreases faster in the first part C M′ of the 2d quadrant C B, than when it arrives near the end of it, at B.

3. The tangent A T becomes negative as often as it meets the radius O M produced, on the opposite side of the point A, or diameter A B, from that in which it is first drawn; and as this takes place both when the arc A M becomes greater than A C, and when, by its further increase, it is greater than A C B D, it follows that the tangents of all arcs in the 1st and 3d quadrants A C, B D are positive, or +, and that the tangents of all arcs in the 2d and 4th quadrants C B, D A are negative, or —.

In the 1st quadrant A C, the tangent A T increases from o till it becomes infinite, or greater than any given quantity; and during the 2d quadrant C B, it decreases negatively, from an infinite quantity to o. After this, it is again affirmative in the 3d quadrant B D, and increases from o to infinity, as in the 1st quadrant; and in the 4th quadrant D A, it decreases from an infinitely negative quantity to o, as in the 2d quadrant.

It is also apparent, from the nature of the figure, that the tangent A T increases more slowly about the middle of the quadrant A C than in any other part of it; and that it augments with great rapidity as the point M approaches near C; having no limit to its increase, as the sine has, but admitting of all possible degrees of magnitude, from o to infinity.

4. Again, as the cotangent C S is computed from the point C, in the same manner as the tangent A T is computed from A, it will evidently vary in its direction and length like the latter, being + in the 1st and 3d quadrants A C, B D, and — in the 2d and 4th C B, D A. In the 1st quadrant A C, it decreases from infinity to o; and in the 2d quadrant C B, it increases negatively from o to infinity. After this, it becomes affirmative in the 3d quadrant B D, and decreases from infinity to o; and in the 4th quadrant D A, it increases negatively from o to infinity, as in the 2d quadrant.

5. In like manner, the secant becomes — as often as the revolving radius O M passes the centre O, and changes with the cosine, being + in the 1st and 4th quadrants A C, D A, and — in the 2d and 3d C B, B D. In the 1st quadrant A C it increases from radius to infinity; and in the 2d quadrant C B, it decreases, negatively, from an infinite quantity to radius. After this, it increases negatively in the 3d quadrant B D from radius to infinity; and in the 4th quadrant D A it is again affirmative, and decreases from infinity to radius, as in the 1st quadrant.

6. In the same way it may also be shown, that the

cosecant O T changes with the sine, being + in the 1st and 2d quadrants, A C, C B, and — in the 3d and 4th D B, D A. In the first quadrant A C, it decreases from infinity to radius, and in the 2d quadrant C B, it increases from radius to infinity. After this it decreases negatively in the 3d quadrant B D, from infinity to radius; and in the 4th quadrant D A, it again increases negatively till it becomes infinite.

7. From an inspection of the figure, it likewise appears, that the versed sine A P increases from o during the 1st semicircle A C B, till it becomes equal to the diameter A B, which is its utmost limit. It then decreases during the 2d semicircle B D A, till it becomes o; but being always computed in the same direction, from A towards B, it is positive, or +, in every part of the circumference. The same also takes place with respect to the chords A M, M M, &c. each of them being common to two arcs, which are, together, equal to the whole circle.

8. It may, also, be further observed, by way of elucidation, that all these lines change their directions as often as they become either infinite or nothing. When they become infinite, their increase is at its utmost limit; after which they take a contrary direction, and decrease. When they become o, their decrease is at its utmost limit; after which they again increase in a contrary direction; thus changing their algebraic signs whenever they pass through a state of infinity or a state of nothingness.

9. These changes, in the signs of the several trigonometrical lines, may be commodiously exhibited, at one view, as in the following table:

	1st quad.	2d quad.	3d quad.	4th quad.
Sin and cosec	+	+	−	−
Cos and sec	+	−	−	+
Tan and cot	+	−	+	−

10. The value of these lines, at the termination of each quadrant of the circle, may also be exhibited in a similar manner, as below:

	0°	90°	180°	270°	360°
Sin	0	r	0	$-r$	0
Cos	r	0	$-r$	0	r
Tan	0	∞	0	∞	0
Cot	∞	0	∞	0	∞
Sec	r	∞	$-r$	∞	r
Cosec	∞	r	∞	$-r$	∞

11. Beside the limits here mentioned, it is common, in many analytical processes, to employ, indifferently, arcs of all magnitudes, whether negative or positive, or greater or less than 360°; in which cases their sines, cosines, &c. may be derived from the above figure, in nearly the same way with the former.

Thus, if to any arc A M, there be added one or more circumferences of the circle, it is evident that they will terminate again exactly in the point M, and that the arc, so augmented, will have the same positive or negative sine, cosine, &c. with the arc A M. Whence, if c denote an entire circumference, or 360°, we shall have

$$\begin{array}{l}
\text{Sin } x = \sin (c + x) = \sin (2c + x) = \sin (3c + x) \text{ \&c.} \\
\cos x = \cos (c + x) = \cos (2c + x) = \cos (3c + x) \text{ \&c.} \\
\tan x = \tan (c + x) = \tan (2c + x) = \tan (3c + x) \text{ \&c.} \\
\cot x = \cot (c + x) = \cot (2c + x) = \cot (3c + x) \text{ \&c.} \\
\text{\&c.}
\end{array}$$

12. Also, if two equal arcs, A M, A m be taken in opposite directions on the circumference of the circle A C B D, one being considered as positive, and the other as negative, their sines, in this case, will be equal, but affected with contrary signs, while the cosines will be the same for each. Whence

$$\begin{array}{l}
\text{Sin } (-a) = -\sin a;\ \cos (-a) = +\cos a \\
\text{Tan } (-a) = -\tan a;\ \cot (-a) = -\cot a \\
\text{Sec } (-a) = +\sec a;\ \text{cosec } (-a) = -\text{cosec } a \\
\text{\&c.}
\end{array}$$

13. Also, if π be made to denote the semicircumference of a circle, the radius of which is r, and n be either o, or any whole number, the above table may be rendered general for an arc of any magnitude whatever. Thus,

$$\begin{array}{l|l}
\text{Sin } n\pi = o & \text{Cos } 2n\pi = r \\
\text{Sin } \frac{4n+1}{2}\pi = r & \text{Cos } \frac{4n+1}{2}\pi = o \\
\text{Sin } \frac{4n+2}{2}\pi = o & \text{Cos } \frac{4n+2}{2}\pi = -r \\
\text{Sin } \frac{4n+3}{2}\pi = -r & \text{Cos } \frac{4n+3}{2}\pi = o \\
\text{Sin } \frac{4n+4}{2}\pi = o & \text{Cos } \frac{4n+4}{2}\pi = r \\
\text{Sin } \frac{4n-1}{2}\pi = -r & \text{Cos } \frac{4n-1}{2}\pi = o\ (i).
\end{array}$$

(i) This table is formed by adding the circumference of the circle continually to the values of the quadrants in the former table.

And $\sin a = \sin (\pi - a) = \sin (2\pi - a) = \sin (3\pi - a)$ &c.

14. It is likewise evident, from what has been said, that if a single trigonometrical line only be given, it can always be placed in two different quarters of the circle, with its proper sign; but if two of these lines be given, which are not the reciprocals of each other, there will be only one quadrant in which both their signs will agree.

Thus, if the tangent of any arc or angle be expressed by the form $\frac{a}{b}$, the numerator a may be considered as representing a sine, and the denominator b a cosine; and the union of the two signs determines the quadrant in which the arc or angle must be placed: as, for instance, $\frac{+a}{+b}$ belongs to the 1st quadrant, $\frac{+a}{-b}$ to the 2d quadrant, $\frac{-a}{-b}$ to the 3d quadrant, and $\frac{-a}{+b}$ to the 4th quadrant.

15. In addition to these observations, it may also be remarked, that the sine, tangent, &c. of any arc being of the same magnitude as the cosine, cotangent, &c. of its complement, and vice versâ, the values of these lines may be expressed in terms of each other, as follows:

$$\text{Sin } a = \cos (90° - a), \ \cos a = \sin (90° - a)$$
$$\tan a = \cot (90° - a), \ \cot a = \tan (90° - a)$$
&c.

16. Moreover, as the sine, cosine, &c. of any arc, is of the same magnitude as the sine, cosine, &c. of its supplement, the same lines may be expressed, in a similar manner, with their proper signs, thus:

$$\text{Sin } a = \sin(180^\circ - a),\ \cos a = -\cos(180^\circ - a)$$
$$\tan a = -\tan(180^\circ - a),\ \cot a = -\cot(180^\circ - a)$$

&c.

17. Or, by substituting $90^\circ - a$ for a, in each of the latter forms, it will appear that the sine, cosine, &c. of any arc or angle below 90°, is equal to the sine, cosine, &c. of an arc or angle as much above 90° as the other is less. Thus,

$$\text{Sin } (90^\circ - a) = \sin(90^\circ + a);\ \cos(90^\circ - a) = -\cos(90^\circ + a).$$

$$\text{Tan } (90^\circ - a) = -\tan(90^\circ + a);\ \cot(90^\circ + a) = -\cot(90^\circ - a)$$

&c.

In each of these forms, however, regard must be had to the change of signs, when the arc or angle a is greater than 90°, which may be easily done: for since $\cos a = \sin(90^\circ - a)$, it is plain, that knowing how to value the sines in all possible cases, we shall be able to value the cosine, and thence all the rest of the trigonometrical lines.

OF TRIGONOMETRICAL FORMULÆ.

18. From the figure above given, it will be easy, by means of right-angled and similar triangles, to deduce the following formulæ, for the sine, cosine, tangent, &c. of any arc or angle in terms of the rest. Thus,

$$\text{Sin } a = \sqrt{r^2 - \cos^2 a} = \frac{r \tan a}{\sqrt{r^2 + \tan^2 a}} = \frac{r^2}{\sqrt{r^2 + \cot^2 a}}$$
$$= \frac{r\sqrt{\sec^2 a - r^2}}{\sec a} = \frac{r^2}{\operatorname{cosec} a} = \frac{\cos a \tan a}{r} = \frac{r \cos a}{\cot a}$$
$$= \frac{\cos a \sec a}{\operatorname{cosec} a} = \frac{r \tan a}{\sec a} = \frac{\tan a \cot a}{\operatorname{cosec} a}.$$

$$\text{Cos } a = \sqrt{r^2 - \sin^2 a} = \frac{r^2}{\sqrt{r^2 + \tan^2 a}} = \frac{r \cot a}{\sqrt{r^2 + \cot^2 a}}$$
$$= \frac{r^2}{\sec a} = \frac{r\sqrt{\operatorname{cosec}^2 a - r^2}}{\operatorname{cosec} a} = \frac{r \sin a}{\tan a} = \frac{\sin a \cot a}{r}$$
$$= \frac{\tan a \cot a}{\sec a} = \frac{\sin a \operatorname{cosec} a}{\sec a}.$$

$$\text{Tan } a = \frac{r^2}{\cot a} = \frac{r \sin a}{\sqrt{r^2 - \sin^2 a}} = \frac{r\sqrt{r^2 - \cos^2 a}}{\cos a} = \sqrt{\sec^2 a - r^2}$$
$$= \frac{r^2}{\sqrt{\operatorname{cosec}^2 a - r^2}} = \frac{r \sin a}{\cos a} = \frac{r^3 \cos a}{\sin a \cot^2 a} = \frac{\cos a \sec a}{\cot a}$$
$$= \frac{r \sec a}{\operatorname{cosec} a} = \frac{\sin a \operatorname{cosec} a}{\cot a}.$$

$$\text{Cot } a = \frac{r^2}{\tan a} = \frac{\sqrt{r^2 - \sin^2 a}}{\sin a} = \frac{r \cos a}{\sqrt{r^2 - \cos^2 a}} = \frac{r^2}{\sqrt{\sec^2 a - r^2}}$$
$$= \sqrt{\operatorname{cosec}^2 a - r^2} = \frac{r \cos a}{\sin a} = \frac{r^3 \sin a}{\cos a \tan^2 a} = \frac{\cos a \sec a}{\tan a}$$
$$= \frac{r \operatorname{cosec} a}{\sec a} = \frac{\sin a \operatorname{cosec} a}{\tan a}.$$

$$\text{Sec } a = \frac{r \operatorname{cosec} a}{\sqrt{\operatorname{cosec}^2 a - r^2}} = \frac{r^2}{\sqrt{r^2 - \sin^2 a}} = \frac{r^2}{\cos a} = \sqrt{r^2 + \tan^2 a}$$
$$= \frac{r\sqrt{r^2 + \cot^2 a}}{\cot a} = \frac{r \tan a}{\sin a} = \frac{\cot a \tan a}{\cos a} = \frac{\sin a \operatorname{cosec} a}{\cos a}$$
$$= \frac{r^3}{\sin a \cot a} = \frac{r \operatorname{cosec} a}{\cot a} = \frac{\tan a \operatorname{cosec} a}{r}.$$

$$\text{Cosec } a = \frac{r \sec a}{\sqrt{\sec^2 a - r^2}} = \frac{r^2}{\sin a} = \frac{r^2}{\sqrt{r^2 - \cos^2 a}} = \frac{r\sqrt{r^2 + \tan^2 a}}{\tan a}$$
$$= \sqrt{r^2 + \cot^2 a} = \frac{r \cot a}{\cos a} = \frac{\tan a \cot a}{\sin a} = \frac{r^3}{\cos a \tan a}$$
$$= \frac{r \sec a}{\tan a} = \frac{\cot a \sec a}{r} = \frac{\cos a \sec a}{\sin a}.$$

19. The versed sines and chords, being seldom used, are omitted in the above table; but if, in any case, they should be wanted, they may be readily expressed in terms of the rest, by substituting the particular values of them, given below, in any of those forms.

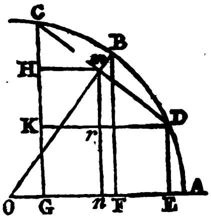
Vers $a = r - \cos a$, covers $a = r - \sin a$, supvers $a = r + \cos a$.

Ch $a = \sqrt{2r(r - \cos a)}$, coch $a = \sqrt{2r(r - \sin a)}$ supch $a = \sqrt{2r(r + \cos a)}$.

In all of which forms regard must be had to the change that takes place in the signs, when arc a is greater than 90°.

Having thus exhibited the various expressions of the sine, tangent, &c. of a single arc, it will now be proper to show the method of obtaining the sine and cosine of the sum and difference of any two arcs, upon which the greater part of the most useful properties, both of these and the other trigonometrical lines, entirely depend.

20. For this purpose, let A B, the greater of two proposed arcs, be denoted by a, and B C, the less, by b: also, make B D equal to B C; and having joined C D, O B, let fall the perpendiculars C G, mn, B F, D E, and draw m H, D K parallel to the radius O A.



Then, because the chord C D is bisected in m, H C will be equal to H K, or $m\,r$, and D r to r K: also, m H is equal to G n, or n E, and $m\,n$ to G H.

And since Cmo, Hmn are right angles, if the angle

$\mathrm{H}m\mathrm{O}$, which is common, be taken away, the remaining angle $\mathrm{C}m\mathrm{H}$ will be equal to $\mathrm{O}mn$, or OBF.

Hence, the triangles $\mathrm{CH}m$, $\mathrm{O}nm$, and OFB being similar, we shall have

$$\left\{\begin{array}{l}\mathrm{OB}:\mathrm{O}m::\mathrm{BF}:mn\\ \mathrm{OB}:\mathrm{OF}::\mathrm{C}m:\mathrm{HC}\end{array}\right\}\text{or}\left\{\begin{array}{l}mn=\dfrac{\sin a\cos b}{r}\\ \mathrm{HC}=\dfrac{\sin b\cos a}{r}\end{array}\right.$$

But $mn+\mathrm{HC}=\mathrm{GH}+\mathrm{HC}=\mathrm{CG}$, which is the sine of AC, or of $a+b$; and $mn-\mathrm{HC}=\mathrm{GH}-\mathrm{HK}=\mathrm{KG}=\mathrm{DE}$, which is the sine of AD, or of $a-b$. Whence

$$\operatorname{Sin}(a+b)=\frac{\sin a\cos b+\sin b\cos a}{r}$$

$$\operatorname{Sin}(a-b)=\frac{\sin a\cos b-\sin b\cos a}{r}$$

Again,

$$\left\{\begin{array}{l}\mathrm{OB}:\mathrm{OF}::\mathrm{O}m:\mathrm{O}n\\ \mathrm{OB}:\mathrm{BF}::\mathrm{C}m:m\mathrm{H}\end{array}\right\}\text{or}\left\{\begin{array}{l}\mathrm{O}n=\dfrac{\cos a\cos b}{r}\\ m\mathrm{H}=\dfrac{\sin a\sin b}{r}\end{array}\right.$$

But $\mathrm{O}n-m\mathrm{H}=\mathrm{O}n-\mathrm{G}n=\mathrm{OG}$, which is the cosine of AC, or of $a+b$; and $\mathrm{O}n+m\mathrm{H}=\mathrm{O}n+\mathrm{G}n=\mathrm{O}n+n\mathrm{E}=\mathrm{OE}$, which is the cosine of AD, or of $a-b$. Whence, also

$$\operatorname{Cos}(a+b)=\frac{\cos a\cos b-\sin a\sin b}{r}$$

$$\operatorname{Cos}(a-b)=\frac{\cos a\cos b+\sin a\sin b}{r}\qquad\text{Q. E. I.}$$

21. From these expressions for the sine and cosine of the sum and difference of any two arcs, and the values of the simple arcs, given in the preceding table, the formulæ for their tangents, cotangents, &c. may be readily obtained, as follows:

$$\text{Tan}\quad (a \pm b) = \frac{r^2 (\tan a \pm \tan b)}{r^2 \mp \tan a \tan b}$$

$$\text{Cot}\quad (a \pm b) = \frac{\cot a \cot b \mp r^2}{\cot b \pm \cot a}$$

$$\text{Sec}\quad (a \pm a) = \frac{r \sec a \sec b}{r^2 \mp \tan a \tan b}$$

$$\text{Cosec}\ (a \pm a) = \frac{\text{cosec}\, a\ \text{cosec}\, b}{\cot b \pm \cot a}$$

The formulæ for the chords and versed sines are, also, as below:

$$\text{Ch}\ (a \pm b) = \frac{\text{ch}\ a\ \text{supch}\ b \pm \text{ch}\ b\ \text{supch}\ a}{2\,r}$$

$$\text{Vers}\ (a \pm b) = \frac{(\sqrt{\text{vers}\ a\ \text{supvers}\ b} \pm \sqrt{\text{vers}\ b\ \text{subvers}\ a})^2}{2\,r}$$

22. Hence, if π be made to represent the semicircumference of a circle, as before, and $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, &c. be substituted for a, and a for b, in the expressions above given, for the sine and cosine of the sum and difference of any two arcs, we shall have

$$\begin{array}{l|l}
\text{Sin}\ (\tfrac{1}{2}\pi + a) = +\cos a & \text{Sin}\ (\tfrac{1}{2}\pi - a) = +\cos a \\
\text{Cos}\ (\tfrac{1}{2}\pi + a) = -\sin a & \text{Cos}\ (\tfrac{1}{2}\pi - a) = +\sin a \\
\text{Sin}\ (\pi + a) = -\sin a & \text{Sin}\ (\pi - a) = +\sin a \\
\text{Cos}\ (\pi + a) = -\cos a & \text{Cos}\ (\pi - a) = -\cos a \\
\text{Sin}\ (\tfrac{3}{2}\pi + a) = -\cos a & \text{Sin}\ (\tfrac{3}{2}\pi - a) = -\cos a \\
\text{Cos}\ (\tfrac{3}{2}\pi + a) = +\sin a & \text{Cos}\ (\tfrac{3}{2}\pi - a) = -\sin a \\
\text{Sin}\ (2\pi + a) = +\sin a & \text{Sin}\ (2\pi - a) = -\sin a \\
\text{Cos}\ (2\pi + a) = +\cos a & \text{Cos}\ (2\pi - a) = +\cos a
\end{array}$$

23. Or, if n be zero, or any positive or negative whole number whatever, the same formulæ may be rendered more general, as follows:

$\text{Sin}\left(\frac{4n+1}{2}\pi+a\right)=+\cos a$	$\text{Sin}\left(\frac{4n+1}{2}\pi-a\right)=+\cos a$
$\text{Cos}\left(\frac{4n+1}{2}\pi+a\right)=-\sin a$	$\text{Cos}\left(\frac{4n+1}{2}\pi-a\right)=+\sin a$
$\text{Sin}\left(\frac{4n+2}{2}\pi+a\right)=-\sin a$	$\text{Sin}\left(\frac{4n+2}{2}\pi-a\right)=+\sin a$
$\text{Cos}\left(\frac{4n+2}{2}\pi+a\right)=-\cos a$	$\text{Cos}\left(\frac{4n+2}{2}\pi-a\right)=-\cos a$
$\text{Sin}\left(\frac{4n+3}{2}\pi+a\right)=-\cos a$	$\text{Sin}\left(\frac{4n+3}{2}\pi-a\right)=-\cos a$
$\text{Cos}\left(\frac{4n+3}{2}\pi+a\right)=+\sin a$	$\text{Cos}\left(\frac{4n+3}{2}\pi-a\right)=-\sin a$
$\text{Sin}\left(\frac{4n+4}{2}\pi+a\right)=+\sin a$	$\text{Sin}\left(\frac{4n+4}{2}\pi-a\right)=-\sin a$
$\text{Cos}\left(\frac{4n+4}{2}\pi+a\right)=+\cos a$	$\text{Cos}\left(\frac{4n+4}{2}\pi-a\right)=+\cos a$
$\text{Sin}\left(\frac{4n+1}{2}\pi\pm a\right)=\mp\cos a$	$\text{Cos}\left(\frac{4n-1}{2}\pi\pm a\right)=\pm\sin a$

24. From the expressions for the sine, cosine, &c. of the sum of two arcs, we may also easily deduce the following formulæ for the sine, cosine, &c. of the double arc, by barely taking b equal to a, and then substituting the values of the lines, thus obtained, in terms of the rest. Thus,

$$\text{Sin } 2\,a = \frac{2\sin a\cos a}{r} = \frac{2\sin^2 a}{\tan a} = \frac{2\cos^2 a}{\cot a} = \frac{2\,r^2\tan a}{r^2+\tan^2 a}$$

$$= \frac{2\,r^2}{\tan a+\cot a} = \frac{2\,r^3\cot a}{r^2+\cot^2 a} = \frac{2\,r\sin a}{\sec a} = \frac{2\,r^2\tan a}{\sec^3 a}$$

$$= \frac{2\,r^3\cot a}{\text{cosec}^2\,a} = \tfrac{1}{2}\,\text{ch}\;4\,a$$

$$\text{Cos } 2\,a = \frac{\cos^2 a-\sin^2 a}{r} = \frac{r^2-2\sin^2 a}{r} = \frac{2\cos^2 a-r^2}{r}$$

$$= \frac{r(r^2-\tan^2 a)}{r^2+\tan^2 a} = \frac{r\,(\cot^2 a-r^2)}{\cot^2 a+r^2} = \frac{r\,(\cot a-\tan a)}{\cot a+\tan a}$$

$$= \frac{r\,(2\,r^2-\sec^2 a)}{\sec^2 a} = \frac{r\,(\text{cosec}^2 a-2\,r^2)}{\text{cosec}^2\,a} = \tfrac{1}{2}\,\text{supch}\;4a$$

$$\text{Tan } 2\,a = \frac{2\,r^2 \tan a}{r^2 - \tan^2 a} = \frac{2\,r^2}{\cot a - \tan a} = \frac{2 r \cos a \sin a}{r^2 - 2 \sin^2 a} = \frac{2 r \sin a \cos a}{2 \cos^2 a - r^2}$$

$$= \frac{2\,r^2 \cot a}{\cot^2 a - r^2} = \frac{2\,r^2 \tan a}{2\,r^2 - \sec^2 a} = \frac{2\,r^2 \cot a}{\operatorname{cosec}^2 a - 2\,r^2}.$$

$$\text{Cot } 2\,a = \frac{\cot^2 a - r^2}{2 \cot a} = \frac{r^2 - \tan^2 a}{2 \tan a} = \frac{\cot a - \tan a}{2} = \frac{r\,(r^2 - 2 \sin^2 a)}{2 \sin a \cos a}$$

$$= \frac{r\,(2 \cos^2 a - r^2)}{2 \sin a \cos a} = \frac{2\,r^2 - \sec^2 a}{2 \tan a} = \frac{\operatorname{cosec}^2 a - 2\,r^2}{2 \cot a}.$$

$$\text{Sec } 2\,a = \frac{r \sec^2 a}{2\,r^2 - \sec^2 a} = \frac{r \operatorname{cosec}^2 a}{\operatorname{cosec}^2 a - 2\,r^2} = \frac{r^3}{r^2 - 2 \sin^2 a}$$

$$= \frac{r^3}{2 \cos^2 a - r^2} = \frac{r\,(r^2 + \tan^2 a)}{r^2 - \tan^2 a} = \frac{r\,(\cot^2 a + r^2)}{\cot^2 a - r^2}$$

$$= \frac{r\,(\cot a + \tan a)}{\cot a - \tan a}.$$

$$\text{Cosec } 2\,a = \frac{r^3}{2 \sin a \cos a} = \frac{r^2 + \tan^2 a}{2 \tan a} = \frac{\cot a + \tan a}{2}$$

$$= \frac{r^2 + \cot^2 a}{2 \cot a} = \frac{r \sec a}{2 \sin a} = \frac{\operatorname{cosec}^2 a}{2 \cot a} = \frac{\sec^2 a}{2 \tan a}$$

$$= \frac{\sec a \operatorname{cosec} a}{2\,r} = \frac{r \operatorname{cosec} a}{2 \cos a}.$$

25. The versed sines and chords of the double arcs may also be expressed in terms of the rest, by substituting the following particular values of them in any of the above forms.

Vers $2\,a = \frac{2 \sin^2 a}{r}$, covers $2\,a = \frac{r^2 - 2 \sin a \cos a}{r}$ $= \frac{2 \sin^2 (45° - a)}{r}$, supvers $2\,a = \frac{2 \cos^2 a}{r}$.

Ch $2\,a = 2 \sin a$, coch $2\,a = 2 \sin (45° - a)$, supch $2\,a = 2 \cos a$.

26. In like manner, if $\frac{1}{2}\,a$ be substituted for a, in each of the above expressions, for the sine, cosine, &c. of the double arc, we shall obtain the following for-

mulæ for the sine, cosine, &c. of the single arc, in terms of the sine, cosine, &c. of the half arc.

$$\text{Sin } a = \frac{2\sin\frac{1}{2}a\cos\frac{1}{2}a}{r} = \frac{2\sin^2\frac{1}{2}a}{\tan\frac{1}{2}a} = \frac{2\cos^2\frac{1}{2}a}{\cot\frac{1}{2}a} = \frac{2r^2\tan\frac{1}{2}a}{r^2+\tan^2\frac{1}{2}a}$$

$$= \frac{2r^2}{\tan\frac{1}{2}a+\cot\frac{1}{2}a} = \frac{2r^2\cot\frac{1}{2}a}{r^2+\cot^2\frac{1}{2}a} = \frac{2r\sin\frac{1}{2}a}{\sec\frac{1}{2}a} = \frac{2r^2\tan\frac{1}{2}a}{\sec^2\frac{1}{2}a}$$

$$= \frac{2r^2\cot\frac{1}{2}a}{\text{cosec}^2\frac{1}{2}a} = \frac{2\tan a\cos^3\frac{1}{2}a}{r^2} = \frac{2r\cos\frac{1}{2}a}{\text{cosec}\frac{1}{2}a} = \tfrac{1}{2}\text{ch } 2a.$$

$$\text{Cos } a = \frac{\cos^2\frac{1}{2}a-\sin^2\frac{1}{2}a}{r} = \frac{r^2-2\sin^2\frac{1}{2}a}{r} = \frac{2\cos^2\frac{1}{2}a-r^2}{r}$$

$$= \frac{r(r^2-\tan^2\frac{1}{2}a)}{r^2+\tan^2\frac{1}{2}a} = \frac{r(\cot^2\frac{1}{2}a-r^2)}{\cot^2\frac{1}{2}a+r^2} = \frac{r(\cot\frac{1}{2}a-\tan\frac{1}{2}a)}{\cot\frac{1}{2}a+\tan\frac{1}{2}a}$$

$$= \frac{r(2r^2-\sec^2\frac{1}{2}a)}{\sec^2\frac{1}{2}a} = \frac{r(\text{cosec}^2\frac{1}{2}a-2r^2)}{\text{cosec}^2\frac{1}{2}a} = \tfrac{1}{2}\text{supch } 2a.$$

$$\text{Tan } a = \frac{2r^2\tan\frac{1}{2}a}{r^2-\tan^2\frac{1}{2}a} = \frac{2r^2}{\cot\frac{1}{2}a-\tan\frac{1}{2}a} = \frac{2r\sin\frac{1}{2}a\cos\frac{1}{2}a}{r^2-2\sin^2\frac{1}{2}a}$$

$$= \frac{2r\sin\frac{1}{2}a\cos\frac{1}{2}a}{2\cos^2\frac{1}{2}a-r^2} = \frac{2r^2\cot\frac{1}{2}a}{\cot^2\frac{1}{2}a-r^2} = \frac{2r^2\tan\frac{1}{2}a}{2r^2-\sec^2\frac{1}{2}a}$$

$$= \frac{2r^2\cot\frac{1}{2}a}{\text{cosec}^2\frac{1}{2}a-2r^2}.$$

$$\text{Cot } a = \frac{\cot^2\frac{1}{2}a-r^2}{2\cot\frac{1}{2}a} = \frac{r^2-\tan^2\frac{1}{2}a}{2\tan\frac{1}{2}a} = \frac{\cot\frac{1}{2}a-\tan\frac{1}{2}a}{2}$$

$$= \frac{r(r^2-2\sin^2\frac{1}{2}a)}{2\sin\frac{1}{2}a\cos\frac{1}{2}a} = \frac{r(2\cos^2\frac{1}{2}a-r^2)}{2\sin\frac{1}{2}a\cos\frac{1}{2}a} = \frac{2r^2-\sec^2\frac{1}{2}a}{2\tan\frac{1}{2}a}$$

$$= \frac{\text{cosec}^2\frac{1}{2}a-2r^2}{2\cot\frac{1}{2}a}.$$

$$\text{Sec } a = \frac{r\sec^2\frac{1}{2}a}{2r^2-\sec^2\frac{1}{2}a} = \frac{r\,\text{cosec}^2\frac{1}{2}a}{\text{cosec}^2\frac{1}{2}a-2r^2} = \frac{r^3}{r^2-2\sin^2\frac{1}{2}a}$$

$$= \frac{r^3}{2\cos^2\frac{1}{2}a-r^2} = \frac{r(r^2+\tan^2\frac{1}{2}a)}{r^2-\tan^2\frac{1}{2}a} = \frac{r(\cot^2\frac{1}{2}a+r^2)}{\cot^2\frac{1}{2}a-r^2}$$

$$= \frac{r(\cot\frac{1}{2}a+\tan\frac{1}{2}a)}{\cot\frac{1}{2}a-\tan\frac{1}{2}a}.$$

$$\text{Cosec } a = \frac{r^2}{2 \sin \frac{1}{2} a \cos \frac{1}{2} a} = \frac{r^2 + \tan^2 \frac{1}{2} a}{2 \tan \frac{1}{2} a} = \frac{\cot \frac{1}{2} a + \tan \frac{1}{2} a}{2}$$
$$= \frac{r^2 + \cot^2 \frac{1}{2} a}{2 \cot \frac{1}{2} a} = \frac{r \sec \frac{1}{2} a}{2 \sin \frac{1}{2} a} = \frac{\text{cosec}^2 \frac{1}{2} a}{2 \cot \frac{1}{2} a} = \frac{\sec^2 \frac{1}{2} a}{2 \tan \frac{1}{2} a}$$
$$= \frac{\sec \frac{1}{2} a \text{ cosec } \frac{1}{2} a}{2r} = \frac{r \text{ cosec } \frac{1}{2} a}{2 \cos \frac{1}{2} a}$$

Also, vers $a = \frac{2 \sin^2 \frac{1}{2} a}{r}$ covers $a = \frac{2}{r} \sin^2 (45^\circ - \frac{1}{2} a)$

supvers $a = \frac{2 \cos^2 \frac{1}{2} a}{r}$

Ch $a = 2 \sin \frac{1}{2} a$, coch $a = 2 \sin (45^\circ - \frac{1}{2} a)$, supch $a = 2 \cos \frac{1}{2} a$.

27. And by finding the values of the sine, cosine, &c. of $\frac{1}{2} a$ in the most commodious of these latter equations, we shall obtain the sine, cosine, &c. of the half arc in terms of the sine, cosine, &c. of the whole arc.

Thus,

$$\text{Sin } \tfrac{1}{2} a = \tfrac{1}{2}\sqrt{r^2 + r \sin a} - \tfrac{1}{2}\sqrt{r^2 - r \sin a} = r\sqrt{\frac{r - \cos a}{2r}}$$
$$= r\sqrt{\frac{\sec a - r}{2 \sec a}} = \sqrt{\frac{r \text{ vers } a}{2}} = \tfrac{1}{2} \text{ ch } a.$$

$$\text{Cos } \tfrac{1}{2} a = \tfrac{1}{2}\sqrt{r^2 + r \sin a} + \tfrac{1}{2}\sqrt{r^2 - r \sin a} = r\sqrt{\frac{r + \cos a}{2r}}$$
$$= r\sqrt{\frac{\sec a + r}{2 \sec a}} = \sqrt{\frac{r \text{ supv } a}{2}} = \tfrac{1}{2} \text{ supch } a.$$

$$\text{Tan } \tfrac{1}{2} a = \frac{r \sin a}{r + \cos a} = \frac{r^2 - r \cos a}{\sin a} = r\sqrt{\frac{r - \cos a}{r + \cos a}}$$
$$= r\sqrt{\frac{\sec a - r}{\sec a + r}} = r\sqrt{\frac{\text{vers } a}{2r - \text{vers } a}} = \text{cosec } a - \cot a.$$

$$\text{Cot } \tfrac{1}{2} a = \frac{r \sin a}{r - \cos a} = \frac{r^2 + r \cos a}{\sin a} = r\sqrt{\frac{r + \cos a}{r - \cos a}}$$
$$= r\sqrt{\frac{\sec a + r}{\sec a - r}} = r\sqrt{\frac{2r - \text{vers } a}{\text{vers } a}} = \text{cosec } a + \cot a.$$

$$\text{Sec } \tfrac{1}{2} a = r\sqrt{\frac{2r}{r + \cos a}} = r\sqrt{\frac{2 \tan a}{\sin a + \tan a}} = r\sqrt{\frac{2 \sec a}{r + \sec a}}$$
$$= r\sqrt{\frac{2r}{2r - \text{vers } a}} = r\sqrt{\frac{2r}{\text{supv } a}}$$

$$\text{Cosec } \tfrac{1}{2} a = r\sqrt{\frac{2r}{r - \cos a}} = r\sqrt{\frac{2 \tan a}{\tan a - \sin a}} = r\sqrt{\frac{2 \sec a}{\sec a - r}}$$

$$= r\sqrt{\frac{2r}{\text{vers } a}} = r\sqrt{\frac{2r}{2r - \text{supv } a}}.$$

Also, vers $\frac{1}{2} a = r - \cos \frac{1}{2} a$, covers $\frac{1}{2} a = r - \sin \frac{1}{2} a$, supvers $\frac{1}{2} a = r - \cos \frac{1}{2} a$.

Ch $\frac{1}{2} a = \sqrt{2r(r - \cos \frac{1}{2} a)}$, coch $\frac{1}{2} a = \sqrt{2r(r - \sin \frac{1}{2} a)}$, supch $\frac{1}{2} a = \sqrt{2r(r + \cos \frac{1}{2} a)}$.

28. The formulæ given in art. 20 for the sine, cosine, &c. of the sum or difference of any two arcs, also furnish a number of other useful expressions; among which the following may serve to change the product of two or more sines, cosines, &c. into their sums and differences, or vice versâ.

$$\text{Sin } a \sin b = \tfrac{1}{2} r \cos (a - b) - \tfrac{1}{2} r \cos (a + b)$$

$$\text{Cos } a \cos b = \tfrac{1}{2} r \cos (a - b) + \tfrac{1}{2} r \cos (a + b)$$

$$\text{Sin } a \cos b = \tfrac{1}{2} r \sin (a + b) + \tfrac{1}{2} r \sin (a - b)$$

$$\text{Sin } b \cos a = \tfrac{1}{2} r \sin (a + b) - \tfrac{1}{2} r \sin (a - b).$$

To these we may also subjoin the additional forms,

$$\text{Tan } a \tan b = r^2 \frac{\cos (a \sim b) - \cos (a + b)}{\cos (a + b) + \cos (a \sim b)}$$

$$\text{Cot } a \cot b = r^2 \frac{\cos (a + b) + \cos (a \sim b)}{\cos (a \sim b) - \cos (a + b)}$$

$$\text{Tan } a \cot b = r^2 \frac{\sin (a + b) + \sin (a - b)}{\sin (a + b) - \sin (a - b)}$$

$$\text{Tan } b \cot a = r^2 \frac{\sin (a + b) - \sin (a - b)}{\sin (a + b) + \sin (a - b)}$$

$$\text{Sin } a \sin b = r^2 \frac{\cos^2 (a - b) - \cos^2 (a + b)}{4 \cos a \cos b}$$

$$\text{Tan } a \tan b = r^3 \frac{\cos (a - b) - \cos (a + b)}{2 \cos a \cos b}.$$

$$\text{Sin } a \sin b = \begin{cases} \sin^2 \frac{1}{2}(a+b) - \sin^2 \frac{1}{2}(a-b) \\ \text{or} \\ \cos^2 \frac{1}{2}(a \sim b) - \cos^2 \frac{1}{2}(a+b) \end{cases}$$

$$\text{Cos } a \cos b = \begin{cases} \cos^2 \frac{1}{2}(a \sim b) - \sin^2 \frac{1}{2}(a+b) \\ \text{or} \\ \cos^2 \frac{1}{2}(a+b) - \sin^2 \frac{1}{2}(a-b) \end{cases}$$

29. And if in these formulæ there be substituted $\frac{1}{2}(a+b)$ for a, and $\frac{1}{2}(a-b)$ for b, we shall obtain the following ones, which are often employed in trigonometrical computations for reducing the sum or difference of two factors to their product.

$$\text{Sin } a + \sin b = \frac{2}{r} \sin \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a-b)$$

$$\text{Sin } a - \sin b = \frac{2}{r} \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}(a+b)$$

$$\text{Cos } a + \cos b = \frac{2}{r} \cos \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a-b)$$

$$\text{Cos } b - \cos a = \frac{2}{r} \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b)$$

$$\text{Tan } a + \tan b = \frac{r \sin a}{\cos a} + \frac{r \sin b}{\cos b} = \frac{r^2 \sin(a+b)}{\cos a \cos b}$$

$$\text{Tan } a - \tan b = \frac{r \sin a}{\cos a} - \frac{r \sin b}{\cos b} = \frac{r^2 \sin(a-b)}{\cos a \cos b}$$

$$\text{Cot } a + \cot b = \frac{r \cos a}{\sin a} + \frac{r \cos b}{\sin b} = \frac{r^2 \sin(a+b)}{\sin a \sin b}$$

$$\text{Cot } b - \cot a = \frac{r \cos a}{\sin a} - \frac{r \cos b}{\sin b} = \frac{r^2 \sin(a-b)}{\sin a \sin b}$$

$$\text{Cot } a + \tan b = \frac{r \cos a}{\sin a} + \frac{r \sin b}{\cos b} = \frac{r^2 \cos(a-b)}{\sin a \cos b}$$

$$\text{Cot } a - \tan b = \frac{r \cos a}{\sin a} - \frac{r \sin b}{\cos b} = \frac{r^2 \cos(a+b)}{\sin a \cos b}$$

And if $90^\circ - b$ be substituted for b, we shall have the following formulæ, which will be found applicable to some particular cases of spherical trigonometry, where the common rules would require two analogies.

$$\text{Sin}\, a + \cos b = \frac{2}{r} \sin\left(45° + \frac{a-b}{2}\right) \cos\left(\frac{a+b}{2} \sim 45°\right)$$

$$= \frac{2}{r} \cos\left(45° \sim \frac{a-b}{2}\right) \cos\left(45° \sim \frac{a+b}{2}\right)$$

$$\text{Sin}\, a - \cos b = \frac{2}{r} \cos\left(45° + \frac{a-b}{2}\right) \sin\left(\frac{a+b}{2} - 45°\right)$$

$$= \frac{2}{r} \sin\left(45° - \frac{a-b}{2}\right) \sin\left(\frac{a+b}{2} - 45°\right)$$

$$\text{Sin}\, b + \cos a = \frac{2}{r} \sin\left(\frac{b-a}{2} + 45°\right) \cos\left(\frac{b+a}{2} \sim 45°\right)$$

$$= \frac{2}{r} \cos\left(45° - \frac{a-b}{2}\right) \sin\left(46° - \frac{a+b}{2}\right)$$

$$\text{Sin}\, b - \cos a = \frac{2}{r} \sin\left(\frac{a-b}{2} + 45°\right) \sin\left(\frac{a+b}{2} - 45°\right)$$

$$= \frac{2}{r} \cos\left(45° \sim \frac{a-b}{2}\right) \cos\left(45° \sim \frac{a+b}{2}\right)$$

30. Also, since $\frac{\sin a}{\cos a} = \frac{\tan a}{r} = \frac{r}{\cot a}$, we shall obtain, by division, the following expressions, which will be found convenient in many logarithmic computations.

$$\frac{\text{Sin}\, a + \sin b}{\sin a - \sin b} = \frac{\sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}$$

$$\frac{\text{Cos}\, a + \cos b}{\cos b - \cos a} = \frac{\cos \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}$$

$$\frac{\text{Sin}\, a + \sin b}{\cos a + \cos b} = \frac{\cos b - \cos a}{\sin a - \sin b} = \frac{\tan \frac{1}{2}(a+b)}{r}$$

$$\frac{\text{Sin}\, a + \sin b}{\cos b - \cos a} = \frac{\cos a + \cos b}{\sin a - \sin b} = \frac{\cot \frac{1}{2}(a-b)}{r}$$

$$\frac{\text{Sin}\, a - \sin b}{\cos a + \cos b} = \frac{\cos b - \cos a}{\sin a + \sin b} = \frac{\tan \frac{1}{2}(a-b)}{r}$$

$$\frac{\text{Sin}\, a - \sin b}{\cos b - \cos a} = \frac{\cos a + \cos b}{\sin a + \sin b} = \frac{\cot \frac{1}{2}(a+b)}{r}$$

$$\frac{\text{Tan } a+\tan b}{\tan a-\tan b}=\frac{\cot a+\cot b}{\cot b-\cot a}=\frac{\sin(a+b)}{\sin(a-b)}$$

$$\frac{\text{Tan } a+\tan b}{\cot a+\cot b}=\frac{\tan a-\tan b}{\cot b-\cot a}=\frac{\tan a\tan b}{r^2}$$

$$\frac{\text{Tan } a+\cot b}{\cot a+\tan b}=\frac{\cot b-\tan a}{\cot a-\tan b}=\frac{\tan a\cot b}{r^2}$$

$$\frac{\text{Tan } a+\tan b}{\cot a-\tan b}=\frac{\tan a\tan(a+b)}{r^2}=\frac{\tan(a+b)}{\cot a}$$

$$\frac{\text{Cot } a+\cot b}{\cot a-\tan b}=\frac{\cot b\tan(a+b)}{r^2}=\frac{\tan(a+b)}{\tan b}$$

$$\frac{\text{Tan } a-\tan b}{\cot a+\tan b}=\frac{\tan a\tan(a-b)}{r^2}=\frac{\tan(a-b)}{\cot a}$$

$$\frac{\text{Cot } b-\cot a}{\tan b+\cot a}=\frac{\cot b\tan(a-b)}{r^2}=\frac{\tan(a-b)}{\tan b}$$

$$\frac{\text{Sin } a-\cos b}{\sin a+\cos b}=\frac{\tan\left(45^\circ-\frac{a-b}{2}\right)}{\tan\left(45^\circ-\frac{a+b}{2}\right)}=\frac{\cot\left(\frac{a-b}{2}-45^\circ\right)}{\cot\left(\frac{a+b}{2}-45^\circ\right)}$$

To these may also be subjoined the additional forms,

$$\frac{\text{Sin }(a+b)}{\sin a+\sin b}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)},\text{ or }\sin a+\sin b=\frac{\cos\frac{1}{2}(a-b)\sin(a+b)}{\cos\frac{1}{2}(a+b)}$$

$$\frac{\text{Sin }(a+b)}{\sin a-\sin b}=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)},\text{ or }\sin a-\sin b=\frac{\sin\frac{1}{2}(a-b)\sin(a+b)}{\sin\frac{1}{2}(a+b)}$$

31. And if one of the expressions in art. 29, be multiplied by the other, we shall obtain, after some simple substitutions for the two last cases, the following formulæ for the difference of the squares of the sines, tangents, &c. of any two arcs.

$$\text{Sin}^2 a-\sin^2 b=\sin(a+b)\sin(a-b)$$

$$\text{Cos}^2 b-\cos^2 a=\sin(a+b)\sin(a-b)$$

$$\text{Cos}^2 a-\sin^2 b=\cos(a+b)\cos(a-b)$$

$$\text{Tan}^2 a-\tan^2 b=\frac{r^4\sin(a+b)\sin(a-b)}{\cos^2 a\cos^2 b}.$$

$$\text{Cot}^2 b - \cot^2 a = \frac{r^4 \sin(a+b)\sin(a-b)}{\sin^2 a \sin^2 b}$$

$$\text{Cot}^2 a - \cot^2 b = \frac{r^4 \cos(a+b)\cos(a-b)}{\sin^2 a \cos^2 b}$$

$$r^2 + \tan a \tan b = \frac{\cos(a \sim b)\sec a \sec b}{r}$$

$$r^2 - \tan a \tan b = \frac{\cos(a+b)\sec a \sec b}{r}$$

32. Also, if 45° be put for a, and a for b, in the formulæ for the sine, tangent, &c. of the sum and difference of any two arcs, (art. 20, 21,) we shall have, after a few simple substitutions and reductions, the following expressions for the sine, tangent, &c. of $45° \pm a$, $45° \pm \frac{1}{2}a$, and $30° \pm a$.

$$\text{Sin}(45° + a) = \cos(45° - a) = \frac{\cos a + \sin a}{\sqrt{2}}$$

$$\text{Sin}(45° - a) = \cos(45° + a) = \frac{\cos a - \sin a}{\sqrt{2}}$$

$$\text{Sin}(45° + \tfrac{1}{2}a) = \cos(45° - \tfrac{1}{2}a) = \sqrt{\frac{r(r + \sin a)}{2}}$$

$$\text{Sin}(45° - \tfrac{1}{2}a) = \cos(45° + \tfrac{1}{2}a) = \sqrt{\frac{r(r - \sin a)}{2}}$$

$$\text{Tan}(45° + \tfrac{1}{2}a) = \cot(45° - \tfrac{1}{2}a) = r\sqrt{\frac{r + \sin a}{r - \sin a}}$$

$$\text{Tan}(45° - \tfrac{1}{2}a) = \cot(45° + \tfrac{1}{2}a) = r\sqrt{\frac{r - \sin a}{r + \sin a}}$$

$$\text{Tan}(45° + \tfrac{1}{2}a) = \frac{r(r + \sin a)}{\cos a} = \frac{r\cos a}{r - \sin a} = \frac{r\cot a}{\text{cosec}\, a - r}$$

$$\text{Tan}(45° - \tfrac{1}{2}a) = \frac{r(r - \sin a)}{\cos a} = \frac{r\cos a}{r + \sin a} = \frac{r\cot a}{\text{cosec}\, a + r}$$

$$\text{Tan}(45° + a) = \frac{r(r + \tan a)}{r - \tan a} = r\sqrt{\frac{r + \sin 2a}{r - \sin 2a}}$$

$$\text{Tan}(45° - a) = \frac{r(r - \tan a)}{r + \tan a} = r\sqrt{\frac{r - \sin 2a}{r + \sin 2a}}$$

$$\text{Cot}(45° + a) = \frac{r(r - \tan a)}{r + \tan a} = \frac{r(\cot a - r)}{r + \cot a}$$

$$\text{Cot}(45^\circ - a) = \frac{r\,(r + \tan a)}{r - \tan a} = \frac{r\,(\cot a + r)}{\cot a - r}$$

$$\text{Sin}\,(30^\circ + a) = \cos\,(60^\circ - a) = \frac{\cos a + \sin a \sqrt{3}}{2}$$

$$\text{Sin}\,(30^\circ - a) = \cos\,(60^\circ + a) = \frac{\cos a - \sin a \sqrt{3}}{2}$$

33. From art. 38 and 40, we may also readily obtain the formulæ given below, which have been found of considerable use in the computation of the common trigonometrical tables.

$$\text{Sin}\,(n + 1)\,a = \frac{2 \cos a \sin n\,a - r \sin\,(n - 1)\,a}{r}$$

$$\text{Cos}\,(n + 1)\,a = \frac{2 \cos a \cos n\,a - r \cos\,(n - 1)\,a}{r}$$

Or,

$$\text{Sin}\,n\,a = \frac{r \sin\,(n - 2)\,a + 2 \sin a \cos\,(n - 1)\,a}{r}$$

$$\text{Cos}\,n\,a = \frac{r \cos\,(n - 2)\,a - 2 \sin a \sin\,(n - 1)\,a}{r}\ (k).$$

34. The following expressions may likewise be obtained in nearly a similar way with the former; but, being of less use in their application, they are here given separately.

$$\text{Sin}\,a = \frac{\text{Sin}\,(45^\circ + a) - \sin\,(45^\circ - a)}{\sqrt{2}} = \frac{\cos\,(45^\circ - a) - \cos\,(45^\circ + a)}{\sqrt{2}}$$

$$\text{Sin}\,a = \frac{\text{Sin}\,(30^\circ + a) - \sin\,(30^\circ - a)}{\sqrt{3}} = \frac{\cos\,(60^\circ - a) - \cos\,(60^\circ + a)}{\sqrt{3}}$$

$$\text{Sin}\,a = r\,\frac{r^2 - \tan^2(45^\circ - \frac{1}{2}a)}{r^2 + \tan^2(45^\circ - \frac{1}{2}a)} = r\,\frac{\tan\,(45^\circ + \frac{1}{2}a) - \tan\,(45^\circ - \frac{1}{2}a)}{\tan\,(45^\circ + \frac{1}{2}a) + \tan\,(45^\circ - \frac{1}{2}a)}$$

(k) Here, if a be 1°, and n be put = 2, 3, 4, &c. successively, we shall have

$$\text{Sin}\ 2^\circ = \sin 0^\circ + 2 \sin 1^\circ \cos 1^\circ$$
$$\text{Sin}\ 3^\circ = \sin 1^\circ + 2 \sin 1^\circ \cos 2^\circ$$
$$\text{Sin}\ 4^\circ = \sin 2^\circ + 2 \sin 1^\circ \cos 3^\circ \ \&c.$$

And the cosines of 2°, 3°, 4°, &c. may be expressed in a similar manner.

$$\text{Cos}\,a=\frac{2\,r^2}{\tan(45^\circ+\frac{1}{2}a)+\tan(45^\circ-\frac{1}{2}a)}=\frac{2\,r^2}{\cot(45^\circ-\frac{1}{2}a)+(\cot 45^\circ+\frac{1}{2}a)}$$

$$\text{Cos}\,a=\frac{2\cos(45^\circ+\frac{1}{2}a)\cos(45^\circ-\frac{1}{2}a)}{r}=\frac{2\sin(45^\circ-\frac{1}{2}a)\sin(45^\circ+\frac{1}{2}a)}{r}$$

$$\text{Tan}\,a=\frac{\tan(45^\circ+\frac{1}{2}a)-\tan(45^\circ-\frac{1}{2}a)}{2}=\frac{\cot(45^\circ-\frac{1}{2}a)-\cot(45^\circ+\frac{1}{2}a)}{2}$$

35. Finally, to these may be added the equations given below, which have been found of great service in the computation of the sines, tangents, &c. of the common trigonometrical tables.

$\text{Sin}\,(30^\circ+a)=\cos\,(60^\circ-a)=\cos a-\sin\,(30^\circ-a)$
$\text{Cos}\,(30^\circ+a)=\sin\,(60^\circ-a)=\cos\,(30^\circ-a)-\sin a$
$\text{Sin}\,(60^\circ+a)=\cos\,(30^\circ-a)=\sin\,(60^\circ-a)+\sin a$
$\text{Cos}\,(60^\circ+a)=\sin\,(30^\circ-a)=\cos a-\cos\,(60^\circ-a)$

$$\text{Tan}(30^\circ+a)=\cot\,(60^\circ-a)=\frac{\cot(30^\circ-\frac{1}{2}a)-\tan(30^\circ-\frac{1}{2}a)}{2}$$

$$\text{Tan}(30^\circ-a)=\cot\,(60^\circ+a)=\frac{\cot(30^\circ+\frac{1}{2}a)-\tan(30^\circ+\frac{1}{2}a)}{2}$$

$\text{Tan}\,(45^\circ+a)=\tan\,(45^\circ-a)+2\tan 2\,a=\cot\,(45^\circ+a)+2\tan 2\,a.$

$\text{Sec}\,a=\tan a+\tan(45^\circ-\frac{1}{2}a)=\tan\,(45^\circ+\frac{1}{2}a)-\tan a.$

36. (*l*) Having thus given a variety of the most simple and useful expressions for the sines, tangents, &c. of the sum or difference of any two arcs, it may not be improper to subjoin the following formulæ for

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(*l*) Formulæ of this kind may be, obviously, multiplied without end; but it is conceived that the collection here given will be found to be more complete and methodical than any which has hitherto appeared.

the arcs themselves, in which radius is supposed to be unity, or 1.

Arc tan x + arc tan y = arc tan $\frac{x+y}{1-xy}$

Arc tan x — arc tan y = arc tan $\frac{x-y}{1+xy}$

Arc cot x + arc cot y = arc cot $\frac{xy-1}{x+y}$

Arc cot x — arc cot y = arc cot $\frac{xy+1}{y-x}$

Arc tan $x = \frac{1}{2}$ arc tan $\frac{2x}{1-x^2} = \frac{1}{2}$ arc cot $\frac{1-x^2}{2x}$

Arc cot $x = \frac{1}{2}$ arc cot $\frac{x^2-1}{2x} = \frac{1}{2}$ arc tan $\frac{2x}{x^2-1}$

Arc sec x = arc cos $\frac{1}{x}$; arc cosec x = arc sin $\frac{1}{x}$

Arc vers x = 2 arc sin $\sqrt{\frac{x}{2}}$; arc covers $x = 90^\circ \mp 2$ arc sin $\sqrt{\frac{x}{2}}$; arc chord x = 2 arc sin $\frac{x}{2}$.

Also,

Arc to tan $\frac{1}{2}$ + arc to tan $\frac{1}{3}$ = arc 45°

Arc to tan $\frac{1}{7}$ + 2 arc to tan $\frac{1}{3}$ = arc 45°.

37. It may also be remarked, that, by means of the formulæ here given, and the known values of the sines of 30°, 45°, 60° and 90°, it is easy to obtain the values of the sines, cosines, &c. of a great variety of other arcs, in surd numbers; the most simple and commodious of which are the following:

Sin 0° = cos 90° = 0

Sin $7\frac{1}{2}^\circ$ = cos $82\frac{1}{2}^\circ = \frac{r}{2}\sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{2}}$

Sin 9° = cos $81^\circ = \frac{r}{4}\sqrt{3+\sqrt{5}} - \frac{r}{4}\sqrt{5-\sqrt{5}}$

Sin $11\frac{1}{4}^\circ$ = cos $78\frac{3}{4}^\circ = \frac{r}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$

$$\text{Sin } 15^\circ = \cos 75^\circ = \frac{r}{4}(\sqrt{6}-\sqrt{2})$$

$$\text{Sin } 18^\circ = \cos 72^\circ = \frac{r}{4}(-1+\sqrt{5})$$

$$\text{Sin } 22\tfrac{1}{2}^\circ = \cos 67\tfrac{1}{2}^\circ = \frac{r}{2}\sqrt{2-\sqrt{2}}$$

$$\text{Sin } 27^\circ = \cos 63^\circ = \frac{r}{4}\sqrt{5+\sqrt{5}} - \frac{r}{4}\sqrt{3-\sqrt{5}}$$

$$\text{Sin } 30^\circ = \cos 60^\circ = \frac{r}{2}$$

$$\text{Sin } 33\tfrac{3}{4}^\circ = \cos 56\tfrac{1}{4}^\circ = \frac{r}{2}\sqrt{2-\sqrt{2-\sqrt{2}}}$$

$$\text{Sin } 36^\circ = \cos 54^\circ = \frac{r}{4}\sqrt{10-2\sqrt{5}}$$

$$\text{Sin } 37\tfrac{1}{2}^\circ = \cos 52\tfrac{1}{2}^\circ = \frac{r}{2}\sqrt{\frac{4+\sqrt{2}-\sqrt{6}}{2}}$$

$$\text{Sin } 45^\circ = \cos 45^\circ = \frac{r}{2}\sqrt{2}$$

$$\text{Sin } 52\tfrac{1}{2}^\circ = \cos 37\tfrac{1}{2}^\circ = \frac{r}{2}\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}}$$

$$\text{Sin } 54^\circ = \cos 36^\circ = \frac{r}{4}(1+\sqrt{5})$$

$$\text{Sin } 56\tfrac{1}{4}^\circ = \cos 33\tfrac{3}{4}^\circ = \frac{r}{2}\sqrt{2+\sqrt{2-\sqrt{2}}}$$

$$\text{Sin } 60^\circ = \cos 30^\circ = \frac{r}{2}\sqrt{3}$$

$$\text{Sin } 63^\circ = \cos 27^\circ = \frac{r}{4}\sqrt{5+\sqrt{5}} + \frac{r}{4}\sqrt{3-\sqrt{5}}$$

$$\text{Sin } 67\tfrac{1}{2}^\circ = \cos 22\tfrac{1}{2}^\circ = \frac{r}{2}\sqrt{2+\sqrt{2}}$$

$$\text{Sin } 72^\circ = \cos 18^\circ = \frac{r}{4}\sqrt{10+2\sqrt{5}}$$

$$\text{Sin } 75^\circ = \cos 15^\circ = \frac{r}{4}(\sqrt{6}+\sqrt{2})$$

$$\text{Sin } 78\tfrac{3}{4}^\circ = \cos 11\tfrac{1}{4}^\circ = \frac{r}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\text{Sin } 81^\circ = \cos 9^\circ = \frac{r}{4}\sqrt{3+\sqrt{5}} + \frac{r}{4}\sqrt{5-\sqrt{5}}$$

$$\text{Sin } 82\tfrac{1}{2}^\circ = \cos 7\tfrac{1}{2}^\circ = \frac{r}{2}\sqrt{\frac{4+\sqrt{2}+\sqrt{6}}{2}}$$

$$\text{Sin } 90^\circ = \cos 0 = r \ (m).$$

Similar surd expressions may also be readily found for the tangents, secants, &c. of the same arcs, by means of the equations, $\tan = \frac{r \sin}{\cos}$, $\cot = \frac{r \cos}{\sin}$, $\sec = \frac{r^2}{\cos}$, and $\text{cosec} = \frac{r^2}{\sin}$.

38. Again, if the arc b be taken equal to $2a$, $3a$, $4a$, &c. successively, the sine of any multiple of a single arc may be readily determined, by means of the known formula for the sine of the sum of two arcs, given in art. 20.

(m) Since $\sqrt{3+\sqrt{5}} = \frac{1}{2}\sqrt{10} + \frac{1}{2}\sqrt{2}$, and $\sqrt{3-\sqrt{5}} = \frac{1}{2}\sqrt{10} - \frac{1}{2}\sqrt{2}$, it is plain, from the above table, that, considering the $\sqrt{2}$, $\sqrt{5}$, and $\sqrt{10}$, as known quantities, there will be only four extractions of the square root required to obtain the values of the sines and cosines of all the arcs which are multiples of 9°.

It also appears, from the same table, that $\sin(60+x) = \sin(60-x) + \sin x$; $\sin 54^\circ - \sin 18^\circ = \sin 90^\circ - \sin 30^\circ = \frac{r}{2}$; and $\sin 81^\circ + \sin 27^\circ + \sin 9^\circ = \sin 45^\circ + \sin 63^\circ$. Or, more generally,

$$\text{Sin}(18^\circ + x) + \sin(18^\circ - x) = 2 \sin 18^\circ \cos x$$

$$\text{Sin}(54^\circ + x) + \sin(54^\circ - x) = 2 \sin 54^\circ \cos x$$

And because $\sin 54^\circ - \sin 18^\circ = \frac{1}{2}$, and $\cos x = \sin(90^\circ - x)$ we shall have, $\sin(54^\circ + x) + \sin(54^\circ - x) - \sin(18^\circ + x) - \sin(18^\circ - x) = \sin(90^\circ - x)$. Which formulæ may serve as very useful checks in the calculation of trigonometrical tables.

(*n*) Thus, if $b = 2a$, and radius $= 1$, we shall have $\sin 3a = \sin a \cos 2a + \sin 2a \cos a$; but $\sin a = \frac{\sin 2a}{2 \cos a}$ (art. 24), and $\cos 2a = 2\cos^2 a - 1$; whence $\sin a \cos 2a = \frac{\sin 2a}{2 \cos a}(2\cos^2 a - 1) = \sin 2a \cos a - \frac{\sin 2a}{2 \cos a} = \sin 2a \cos a - \sin a$; which value, being substituted in the first equation, gives

$$\text{Sin } 3a = 2 \cos a \sin 2a - \sin a.$$

And, by following the same mode of investigation, we shall readily obtain the sines of the arcs $4a$, $5a$, &c. in terms of the sines and cosines of the inferior arcs, as below:

$$\begin{aligned}
\text{Sin } a &= \sin a \\
\text{Sin } 2a &= 2 \cos a \sin a \\
\text{Sin } 3a &= 2 \cos a \sin 2a - \sin a \\
\text{Sin } 4a &= 2 \cos a \sin 3a - \sin 2a \\
\text{Sin } 5a &= 2 \cos a \sin 4a - \sin 3a, \text{ \&c.}
\end{aligned}$$

Which series, it is obvious, may be continued at pleasure, without any new calculation, by multiplying twice $\cos a$ by the preceding sine, and then subtracting the sine preceding the last, for the next following one,

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(*n*) When the radius, in any trigonometrical expression, is denoted by 1, instead of r, the latter symbol may be readily introduced into the equation, by joining either the simple letter r, or its powers, to such of the factors as will render them all homogeneous, or of the same dimensions. Thus, if radius $= 1$, and $4 \sin^3 a = 3 \sin^2 a - \cos a + 2$, in which the highest term is of 3 dimensions, the equation, by introducing r, will become $4 \sin^3 a = 3r \sin^2 a - r^2 \cos a + 2r^3$.

Thus,

$$\text{Sin } n a = 2 \cos a \sin (n-1) a - \sin (n-2) a$$

Or,

$$\text{Sin } n a = \sin (n-2) a + 2 \sin a \cos (n-1) a .$$

And by substituting the values of cos a, sin $2 a$, sin $3 a$, &c. as taken in terms of the sine and cosine of the single arc, the same series may be easily varied, as follows:

$\text{Sin } a = \sin a$
$\text{Sin } 2 a = 2 \sin a \sqrt{1-\sin^2 a}$
$\text{Sin } 3 a = 3 \sin a - 4 \sin^3 a$
$\text{Sin } 4 a = (4 \sin a - 8 \sin^3 a) \sqrt{1-\sin^2 a}$
$\text{Sin } 5 a = 5 \sin a - 20 \sin^3 a + 16 \sin^5 a$
$\text{Sin } 6 a = (6 \sin a - 32 \sin^3 a + 32 \sin^5 a) \sqrt{1-\sin^2 a}$
$\text{Sin } 7 a = 7 \sin a - 56 \sin^3 a + 112 \sin^5 a - 64 \sin^7 a$

&c.

Or,

$\text{Sin } a = \sqrt{1-\cos^2 a}$
$\text{Sin } 2 a = 2 \cos a \sqrt{1-\cos^2 a}$
$\text{Sin } 3 a = (4 \cos^2 a - 1) \sqrt{1-\cos^2 a}$
$\text{Sin } 4 a = (8 \cos^3 a - 4 \cos a) \sqrt{1-\cos^2 a}$
$\text{Sin } 5 a = (16 \cos^4 a - 12 \cos^2 a + 1) \sqrt{1-\cos^2 a}$
$\text{Sin } 6 a = (32 \cos^5 a - 32 \cos^3 a + 6 \cos a) \sqrt{1-\cos^2 a}$
$\text{Sin } 7 a = (64 \cos^6 a - 80 \cos^4 a + 24 \cos^2 a - 1) \sqrt{1-\cos^2 a}$

&c.

39. Hence, observing the law of the coefficients and exponents of the several terms in the above expressions, we may readily obtain the following general formulæ, for the sine of any multiple of an arc, in terms of the sine of the simple arc. Thus,

$$\text{Sin}\, n a = n \sin a - \frac{n(n^2-1)}{2.3\, r^2} \sin^3 a + \frac{n(n^2-1)\,(n^2-3^2)}{2.3.4.5\, r^4}$$
$$\sin^5 a - \frac{n(n^2-1)\,(n^2-3^2)\,(n^2-5^2)}{2.3.4.5.6.7\, r^6} \sin^7 a \,\text{-}\,\text{-}\, 2^{n-1} \sin^{n-1} a$$

Or,

$$\text{Sin}\, n a = n \sin a - \frac{n^2-1}{2.3\, r^2} (\text{A}) \sin^2 a - \frac{n^2-3^2}{4.5\, r^2} (\text{B}) \sin^2 a$$
$$- \frac{n^2-5^2}{6.7\, r^2} (\text{C}) \sin^2 a - \frac{n^2-7^2}{8.9\, r^2} (\text{D}) \sin^2 a, \text{ \&c.}$$

Where A, B, C, &c. are the preceding terms with their proper signs; and if n be an odd number, the series will terminate; but otherwise not,

Again, if these two formulæ be divided by the expanded value of $\sqrt{r^2-\sin^2} = r - \frac{\sin a}{2\, r} - \frac{\sin^4 a}{8\, r^3}$ &c, and then multiplied by $\sqrt{r^2-\sin^2 a}$, we shall have

$$\text{Sin}\, n a = \left\{ \frac{n}{r} \sin a - \frac{n(n^2-2^2)}{2.3\, r^3} \sin^3 a + \frac{n(n^2-2^2)(n^2-4^2)}{2.3.4.5\, r^5}\right.$$
$$\sin^5 a - \frac{n(n^2-2^2)\,(n^2-4^2)\,(n^2-6^2)}{2.3.4.5.6.7\, r^7} \sin^7 a \;\text{-----}\; \pm 2^{n-1}$$
$$\left.\sin^{n-1} d \right\} \sqrt{r^2-\sin^2}.$$

Or,

$$\text{Sin}\, n a = \left\{ \frac{n}{r} \sin a - \frac{n^2-2^2}{2.3\, r^2} (\text{A}) \sin^2 a - \frac{n^2-4^2}{4.5\, r^2} (\text{B}) \sin^2 a\right.$$
$$\left.\frac{n^2-6^2}{6.7\, r^2} (\text{C}) \sin^2 a - \frac{n^2-8^2}{8.9\, r^2} (\text{D}) \sin^2 a, \text{ \&c.} \right\} \sqrt{r^2-\sin^2 a},$$

In which case, the series will terminate when n is an even number.

And if $\frac{m}{n}$ be substituted for n, in each of the two formulæ in the first part of this article, we shall have the following series for the sine of any part or submultiple of the arc a.

$$\sin\frac{m}{n}a=\frac{m}{n}\sin a-\frac{m(m^2-n^2)}{2.3\,n^3\,r^2}\sin^3 a+\frac{m(m^2-n^2)(m^2-3^2n^2)}{2.3.4.5.\,n^5\,r^4}$$

$$\sin^5 a-\frac{m(m^2-n^2)\,(m^2-3^2\,n^2)\,(m^2-5^2\,n^2)}{2.3.4.5.6.7\,n^7\,r^6}\sin^7 a,\ \&c.$$

Or,

$$\sin\frac{m}{n}\;a\;=\;\frac{m}{n}\;\sin\;a+\frac{n^2-m^2}{2.3\,n^2r^2}\,(\text{A})\;\sin^2 a+\frac{3^2\,n^2-m^2}{4.5\;n^2\,r^2}$$

$$(\text{B})\;\sin^2 a+\frac{5^2\,n^2-m^2}{6.7\,n^2\,r^2}\;(\text{C})\;\sin^2 a,\ \&c.\ (o).$$

40. Moreover, if we still suppose the arc b successively equal to a, $2\,a$, $3\,a$, $4\,a$, &c. and proceed in a similar manner with the formula for the cosine of the sum of two arcs, given in art. 20, we shall obtain the following expressions:

$\cos a = \cos a$

$\cos 2\,a = 2\cos a\cos a - 1$

$\cos 3\,a = 2\cos a\cos 2\,a - \cos a$

$\cos 4\,a = 2\cos a\cos 3\,a - \cos 2\,a$

$\cos 5\,a = 2\cos a\cos 4\,a - \cos 3\,a$, &c.

Where it is plain, as in the former table of sines, that multiplying any cosine by $2\cos a$, and then subtracting the next preceding cosine, we shall obtain the next following one. Thus,

$$\cos n\,a = 2\cos a\cos(n-1)\,a - \cos(n-2)\,a.$$

Or,

$$\cos n\,a = \cos(n-2)\,a - 2\sin a\sin(n-1)\,a.$$

And by substituting the preceding values of $\cos 2a$, $\cos 3a$, &c. in each of the succeeding forms of the above table, the same series may be varied as follows:

(o) The formulæ here given, for the sine and cosine of any arc or its multiple, are equally applicable to the chord of the arc, and the chord of its supplement, using the diameter, or $2r$, instead of r.

$\text{Cos}\ \ a = \cos a$
$\text{Cos}\ 2\,a = 2 \cos^2 a - 1$
$\text{Cos}\ 3\,a = 4 \cos^3 a - 3 \cos a$
$\text{Cos}\ 4\,a = 8 \cos^4 a - 8 \cos^2 a + 1$
$\text{Cos}\ 5\,a = 16 \cos^5 a - 20 \cos^3 a + 5 \cos a$
$\text{Cos}\ 6\,a = 32 \cos^6 a - 48 \cos^4 a + 18 \cos^2 a - 1$
$\text{Cos}\ 7\,a = 64 \cos^7 a - 112 \cos^5 a + 56 \cos^3 a - 7 \cos a,$
&c. Or,

$\text{Cos}\ \ a = \sqrt{1 - \sin^2 a}$
$\text{Cos}\ 2\,a = 1 - 2 \sin^2 a$
$\text{Cos}\ 3\,a = (1 - 4 \sin^2 a) \sqrt{1 - \sin^2 a}$
$\text{Cos}\ 4\,a = 1 - 8 \sin^2 a + 8 \sin^4 a$
$\text{Cos}\ 5\,a = (1 - 12 \sin^2 a + 16 \sin^4 a) \sqrt{1 - \sin^2 a}$
$\text{Cos}\ 6\,a = 1 - 18 \sin^2 a + 48 \sin^4 a - 32 \sin^6 a$
$\text{Cos}\ 7\,a = (1 - 24 \sin^2 a + 80 \sin^4 a - 64 \sin^6 a) \sqrt{1 - \sin^2 a}$
&c.

41. Hence, observing the law of the coefficients and exponents of the several terms of these expressions, we shall obtain the following general formulæ for the cosine of any multiple of an arc, in terms of the cosine of the simple arc. Thus,

$$\text{Cos}\ n\,a = \frac{2^{n-1}}{r^{n-1}}\Big\{\cos\ a - \frac{n}{2^2} r^2 \cos^{n-2} a + \frac{n(n-3)}{2.2^4} r^4 \cos^{n-4} a - \frac{n(n-4)(n-5)}{2.3.2^6} r^6 \cos^{n-6} a + \frac{n(n-5)(n-6)(n-7)}{2.3.4.2^8} r^8 \cos^{n-8} a,\Big\} \text{ \&c.}$$

Or,

$$\text{Cos}\ n\,a = \frac{2^n \cos^n a}{2r^{n-1}} - \frac{n\,r^2}{2^2 \cos^2 a}\text{A} - \frac{(n-3)\,r^2}{2.2^2 \cos^2 a}\text{B} - \frac{(n-4)\,(n-5)\,r^2}{3.2^2\,(n-3)\cos^2 a}\text{C} - \frac{(n-6)\,(n-7) r^2}{4.2^2\,(n-4)\,\cos\ a}\text{D, \&c.}$$

Which series must be continued to $\frac{n+1}{2}$ terms, when n is an odd number, and to $\frac{n+4}{2}$ when it is even; but the same series will be equally true when n is any fractional number.

42. In like manner, since $\tan 3a = \frac{\tan a + \tan 2a}{1 - \tan a \tan 2a}$ (art. 21), and $\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$ (art. 24), if this latter expression be substituted in the former equation, and the whole be reduced to its most simple terms, we shall have

$$\text{Tan } 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}.$$

And by following the same mode of investigation, we shall readily obtain the tangents of the arcs 4 a, 5 a, &c. in terms of the tangent of the single arc, as given below.

$$\text{Tan } a = \tan a$$

$$\text{Tan } 2a = \frac{2 \tan a}{1 - \tan^2 a}$$

$$\text{Tan } 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}$$

$$\text{Tan } 4a = \frac{4 \tan a - 4 \tan^3 a}{1 - 6 \tan^2 a + \tan^4 a}$$

$$\text{Tan } 5a = \frac{5 \tan a - 10 \tan^3 a + \tan^5 a}{1 - 10 \tan^2 a + 5 \tan^4 a}, \text{ \&c.}$$

43. Hence, also, by observing the law of the coefficients and exponents of these last expressions, it will be easy to deduce the following general formula for the tangent of any multiple of a. Thus, $\tan n a =$

$$\frac{n \tan a - \frac{n(n-1)(n-2)}{2.3\, r^2} \tan^3 a + \frac{n(n-1)(n-2)(n-3)(n-4)}{2.3.4.5\, r^4} \tan^5 a, \text{ \&c.}}{1 - \frac{n(n-1)}{2\, r^2} \tan^2 a + \frac{n(n-1)(n-2)(n-3)}{2.3.4\, r^4} \tan^4 a, \text{ \&c.}}$$

Or, Tan $n\,a =$

$$\frac{n \tan a - \frac{(n-1)(n-2)}{2.3\,r^2}(\text{A}) \tan^3 a + \frac{(n-3)(n-4)}{4.5\,r^2}(\text{B}) \tan^5 a \text{ \&c.}}{1 - \frac{n(n-1)}{2\,r^2} \tan^2 a + \frac{(n-2)(n-3)}{3.4\,r^2}(\text{A}') \tan^4 a, \text{ \&c.}}$$

Where n may be any number, either whole or fractional; but the series can only terminate when n is a whole number.

44. The cotangents of the multiple arcs $2\,a$, $3\,a$, $4\,a$, &c. may also be readily expressed, by substituting $\frac{1}{\cot a}$ in the place of the tangent of a, in the last table; which being done, we shall have

$$\text{Cot } a = \cot a$$

$$\text{Cot } 2\,a = \frac{\cot^2 a - 1}{2 \cot a}$$

$$\text{Cot } 3\,a = \frac{\cot^3 a - 3 \cot a}{3 \cot^2 a - 1}$$

$$\text{Cot } 4\,a = \frac{\cot^4 a - 6 \cot^2 a + 1}{4 \cot^3 a - 4 \cot a}$$

$$\text{Cot } 5\,a = \frac{\cot^5 a - 10 \cot^3 a + 5 \cot a}{5 \cot^4 a - 10 \cot^2 a + 1}, \text{ \&c.}$$

45. Hence, also, it will be easy, either from the above expressions, or from the former series for the tangent, to deduce the following formula for the cotangent of any multiple arc. Thus, cot $n\,a =$

$$\frac{\cot^n a - \frac{n(n-1)}{2} r^2 \cot^{n-2} a + \frac{n(n-1)(n-2)(n-3)}{2.3.4} r^4 \cot^{n-4} a, \text{ \&c.}}{n \cot^{n-1} a - \frac{n(n-1)(n-2)}{2.3} r^2 \cot^{n-3} a + \frac{n(n-1)(n-2)(n-3)(n-4)}{2.3.4.5} r^4 \cot^{n-} a}$$

Or, Cot $n\,a =$

$$\frac{\cot^n a - \frac{n(n-1)r^2}{2 \cot^2 a}\text{A} + \frac{(n-2)(n-3)r^2}{3.4 \cot^4 a}\text{B} - \frac{(n-4)(n-5)r^2}{5.6 \cot^6 a}\text{C}, \text{ \&c.}}{n \cot^{n-1} a - \frac{(n-1)(n-2)r^2}{2.3 \cot^3 a}\text{A}' + \frac{(n-3)(n-4)r^2}{4.5 \cot^5 a}\text{B}' - \frac{(n-5)(n-6)r^2}{6.7 \cot^7 a}\text{C}', \text{ \&c.}}$$

46. In like manner, the secants of the same multiple arcs, $2a$, $3a$, $4a$, &c. may be expressed by substituting $\frac{1}{\sec a}$ for cosine of a, in the former table of those lines; which will give

$$\text{Sec } a = \sec a$$

$$\text{Sec } 2a = \frac{\sec^2 a}{2 - \sec^2 a}$$

$$\text{Sec } 3a = \frac{\sec^3 a}{4 - 3\sec^2 a}$$

$$\text{Sec } 4a = \frac{\sec^4 a}{8 - 8\sec^2 a + \sec^4 a}$$

$$\text{Sec } 5a = \frac{\sec^5 a}{16 - 20\sec^2 a + 5\sec^4 a} \text{ \&c.}$$

Or, generally, $\sec na =$

$$\frac{\sec^n a}{2^{n-1}\left\{r^{n-1} - \frac{nr^{n-3}}{2^2}\sec^2 a + \frac{n(n-3)r^{n-5}}{2.2^4}\sec^4 a - \frac{n(n-4)(n-5)r^{n-7}}{2.3.2^6}\right.}$$

&c.

Also,

$$\text{Vers } na = n^2 \text{ vers } a - \frac{n^2-1}{2.3r}(\text{A}) \text{ vers } a - \frac{n^2-2^2}{3.5r}(\text{B}) \text{ vers } a - \frac{n^2-3^2}{4.7r}(\text{C}) \text{ vers } a - \frac{n^2-4^2}{5.9r}(\text{D}) \text{ vers } a, \text{ \&c.}$$

47. Also, by means of art. 40, it will be easy to find the powers of the sine and cosine of the simple arc, in terms of the sine and cosine of certain multiples of that arc. For, since $2\sin^2 a = 1 - \cos 2a$; $4\sin^3 a = 3\sin a - \sin 3a$; and $8\sin^4 a = \cos 4a + 8\sin^2 a - 1 = \cos 4a - 4\cos 2a + 3$, &c. we shall have

$$\text{Sin } a = \sin a$$

$$2\sin^2 a = 1 - \cos 2a$$

$$4\sin^3 a = 3\sin a - \sin 3a$$

$$8\sin^4 a = 3 - 4\cos 2a + \cos 4a$$

$$16\sin^5 a = 10\sin a - 5\sin 3a + \sin 5a$$

$$32\sin^6 a = 10 - 15\cos 2a + 6\cos 4a - \cos 6a$$

$$64\sin a = 35\sin a - 21\sin 3a + 7\sin 5a - \sin 7a, \text{\&c.}$$

Where, beginning at the last term, the law of the coefficients is manifest, being the same with that of the coefficients of a binomial, raised successively to its several powers, except that the numbers 1, 3, 10, &c. standing by themselves, are only half the coefficients of the corresponding terms in the like powers of the binomial.

48. Hence, we shall have $\sin^n a =$

$$\frac{1}{2^{n-1}}\left\{ \pm \sin n a \mp n \sin (n-2)\, a \pm \frac{n(n-1)}{2} \sin (n-4)\, a,\right.$$

&c. Or, $\sin^n a =$

$$\frac{1}{2^{n-1}}\left\{ \pm \cos n a \mp n \cos (n-2)\, a \pm \frac{n(n-1)}{2} \cos (n-4)\, a\right.$$

In the first of which series, the upper sign must be used when n is an odd number, and equal to $4m + 1$, and the lower sign when n is equal to $4m - 1$, m being any number whatever.

In the second series, the upper sign must be used when n is equal to 4 times any number m, and the lower signs when n is equal to twice any odd number m.

49. Similar formulæ may also be found for the successive powers of the cosine of any simple arc, by applying the expressions in art. 40 in the same manner as above. Thus,

$$\begin{aligned}
\text{Cos}\, a &= \cos a \\
2 \cos^2 a &= 1 + \cos 2a \\
4 \cos^3 a &= 3 \cos a + \cos 3a \\
8 \cos^4 a &= 3 + 4 \cos 2a + \cos 4a \\
16 \cos^5 a &= 10 \cos a + 5 \cos 3a + \cos 5a \\
32 \cos^6 a &= 10 + 15 \cos 2a + 6 \cos 4a + \cos 6a \\
64 \cos^7 a &= 35 \cos a + 21 \cos 3a + 7 \cos 5a + \cos 7a,
\end{aligned}$$

&c. &c.

Where the coefficients observe the same law as in the former table of the sines; and if we regard the last terms of these equations as the first, or take them all backwards, the following general formula will be readily obtained: Cos$^n\,a =$

$$\frac{1}{2^{n-1}}\Big\{\cos na + n\cos(n-2)\,a + \frac{n(n-1)}{2}\cos(n-4)\,a + \frac{n(n-1)(n-2)}{2.3}\cos(n-6)\,a \;-\;-\;-\; \frac{n(n-1)(n-2)\text{ \&c.}}{2.3.4,\text{ \&c.}}$$

$\cos(n-n)\,a$, or $\cos a$, according as n is an even or an odd number.

50. From these latter series, expressions may likewise be derived for determining the value of the sine, cosine, &c. in terms of the arc, and vice versâ; but as the mode of deduction, commonly used for this purpose, is not so clear and satisfactory as could be wished, it will be better to employ the doctrine of fluxions, from which they may be easily obtained, as follows:

Let $z =$ the length of the arc, and $x =$ sine; then, by a known formula,

$$\dot{z} = \frac{r\dot{x}}{\sqrt{r^2-x^2}} = r\dot{x}\Big(\frac{1}{r} + \frac{x^2}{2r^3} + \frac{3x^4}{2.4r^5} + \frac{3.5x^6}{2.4.6r^7} + \frac{3.5.7x^3}{2.4.6.8r^9},\text{ \&c.}\Big) = \dot{x} + \frac{x^2\dot{x}}{2r^2} + \frac{3x^4\dot{x}}{2.4r^4} + \frac{3.5x^6\dot{x}}{2.4.6r^6} + \frac{3.5.7x^8\dot{x}}{2.4.6.8r^8}$$

&c.; the fluent of which is $z = x + \frac{x^3}{2.3r^2} + \frac{3x^5}{2.4.5r^4} + \frac{3.5x^7}{2.4.6.7r^6} + \frac{3.5.7x^9}{2.4.6.8r^8}$, &c. And, by reverting the series, we shall have $x = z - \frac{z^3}{2.3r^2} + \frac{z^5}{2.3.4.5r^4} - \frac{z^7}{2.3.4.5.6.7r^6} +$ &c.

Whence $a =$

$$\left\{\begin{array}{l} \sin a + \frac{\sin^3 a}{2.3\,r^2} + \frac{3\sin^5 a}{2.4.5\,r^4} + \frac{3.5\sin^7 a}{2.4.6.7\,r^6} + \frac{3.5.7\sin^9 a}{2.4.6.8.9\,r^8}\ \&c. \\ \qquad\qquad\qquad\text{Or,} \\ \sin a + \frac{\sin^2 a}{2.3\,r^2}\mathrm{A} + \frac{3^2\sin^2 a}{4.5\,r^2}\mathrm{B} + \frac{5^2\sin^2 a}{6.7\,r^2}\mathrm{C} + \frac{7^2\sin^2 a}{8.9\,r^2}\mathrm{D}\ \&c. \end{array}\right.$$

Sin $a =$

$$\left\{\begin{array}{l} a - \frac{a^3}{2.3\,r^2} + \frac{a^5}{2.3.4.5\,r^4} - \frac{a^7}{2.3.4.5.6.7\,r^6} + \frac{a^9}{2.3.4.5.6.7.8.9\,r^8}\ \&c. \\ \qquad\qquad\qquad\text{Or,} \\ a - \frac{a^2}{2.3\,r^2}\mathrm{A} - \frac{a^2}{4.5\,r^2}\mathrm{B} - \frac{a^2}{6.7\,r^2}\mathrm{C} - \frac{a^2}{8.9\,r^2}\mathrm{D} - \frac{a^2}{10.11\,r^2}\mathrm{E}\ \&c. \end{array}\right.$$

And the same series will equally apply to the chord of any arc a, substituting the chord for the sine, and the diameter d, or $2r$, instead of r.

51. Also, if $90° - a$ be substituted for a, we shall have $90° - a = \sin(90° - a) + \frac{\sin^3(90° - a)}{2.3\,r^2}$ &c.; or $a = 90° - \cos a - \frac{\cos^3 a}{2.3\,r^2}$ &c. And $\cos a = r - \frac{2}{r}\sin^2\frac{1}{2}a$ (art. 26) $= r - \frac{2}{r}\left(\frac{a}{2} - \frac{a^3}{2.3.2^3r^2} + \frac{a^5}{2.3.4.5.2^5r^4}\ \&c.\right)^2 =$ $r - \frac{a^2}{2r} + \frac{a^4}{2.3.4\,r^3} - \frac{a^6}{2.3.4.5.6\,r^5} +$, &c.

Whence $a =$

$$\left\{\begin{array}{l} 90° - \cos a - \frac{\cos^3 a}{2.3\,r^2} - \frac{3\cos^5 a}{2.4.5\,r^4} - \frac{3.5\cos^7 a}{2.4.6.7\,r^6}\ \&c. \\ \qquad\qquad\qquad\text{Or,} \\ \frac{r - \cos a}{1} + \frac{r^3 - \cos^3 a}{2.3\,r^2} + \frac{3}{2.4.5}\left(\frac{r^5 - \cos^5 a}{r^4}\right) + \frac{3.5}{2.4.6.7}\left(\frac{r^7 - \cos^7 a}{r^6}\right)\ \&c. \end{array}\right.$$

Cos $a =$

$$\left\{\begin{array}{l} r - \frac{a^2}{2r} + \frac{a^4}{2.3.4\,r^3} - \frac{a^6}{2.3.4.5.6\,r^5} + \frac{a^8}{2.3.\ldots.8\,r^7}\ \&c. \\ \qquad\qquad\qquad\text{Or,} \\ r - \frac{a^2}{2\,r^2}\mathrm{A} - \frac{a^2}{3.4r^2}\mathrm{B} - \frac{a^2}{5.6r^2}\mathrm{C} - \frac{a^2}{7.8\,r^2}\mathrm{D} - \frac{a^2}{9.10}\mathrm{E}\ \&c. \end{array}\right.$$

52. Again, let z = length of the arc, as before, and t = tangent; then, by a well known formula, $\dot{z} = \frac{r^2\dot{t}}{r^2+t^2} = r^2\dot{t}\left(\frac{1}{r^2} - \frac{t^2}{r^4} + \frac{t^4}{r^6}, \&c.\right) = \dot{t} - \frac{t^2\dot{t}}{r^2} + \frac{t^4\dot{t}}{r^4}$, &c. the fluent of which is $z = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4}$, &c. And by reverting the series we shall have $t = z + \frac{z^3}{3r^2} + \frac{2z^5}{15r^4}$, &c.

Whence

$$a = \tan a - \frac{\tan^3 a}{3r^2} + \frac{\tan^5 a}{5r^4} - \frac{\tan^7 a}{7r^6} + \frac{\tan^9 a}{9r^8} - \frac{\tan^{11} a}{11r^{10}} \&c.$$

$$\text{Tan } a = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} + \frac{1382a^{11}}{155925r^{10}} + \frac{21844a^{13}}{6081075r^{12}} + \frac{929569a^{15}}{638512875r^{14}} \&c.$$

53. And since $\tan a = \frac{r^2}{\cot a}$, or $\cot a = \frac{r^2}{\tan a}$, we shall readily obtain, by means of the above series, the following expressions for the cotangent.

$$a = \frac{r^2}{\cot a} - \frac{r^4}{3\cot^3 a} + \frac{r^6}{5\cot^5 a} - \frac{r^8}{7\cot^7 a} + \frac{r^{10}}{9\cot^9 a} - \frac{r^{12}}{11\cot^2 a} + \&c.$$

$$\text{Cot } a = \frac{r^2}{a} - \frac{a}{3} - \frac{a^3}{45r^2} - \frac{2a^5}{945r^4} - \frac{a^7}{4725r^6} - \frac{2a^9}{93555r^8} - \frac{1382a^{11}}{638512875r^{10}} - \frac{4a^{13}}{18243225r^{12}} \&c.$$

54. Also, because $\sec a = \frac{r^2}{\cos a}$, or $\cos a = \frac{r^2}{\sec a}$, the series before given for the cosine, may be easily converted into the following expressions for the secant:

$$a = 90^\circ - \frac{r^2}{\sec a} - \frac{r^4}{2.3\sec^3 a} - \frac{3r^6}{2.4.5\sec^5 a} - \frac{3.5r^8}{2.4.6.7\sec^7 a}, \&c.$$

Or,

$$a = r\left\{\frac{\sec a - r}{\sec a} + \frac{\sec^3 a - r^3}{2.3\sec^3 a} + \frac{3(\sec^5 a - r^5)}{2.4.5\sec^5 a} + \frac{3.5(\sec^7 a - r^7)}{2.4.6.7\sec^7 a} \&c.\right\}$$

$$\text{Sec } a = r + \frac{a^2}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} + \frac{50521a^{10}}{3628800r^9} + \frac{540553a^{12}}{95800320r^{11}} \text{ \&c.}$$

55. In like manner, because cosec $a = \frac{r^2}{\sin a}$, or $\sin a = \frac{r^2}{\text{cosec } a}$, we may obtain, by means of the series before given for the sine, the following expressions for the cosecant:

$$a = \frac{r^2}{\text{cosec } a} + \frac{r^4}{2.3 \text{ cosec}^3 a} + \frac{3r^6}{2.4.5 \text{ cosec}^5 a} + \frac{3.5r^8}{2.4.6.7 \text{ cosec}^7 a} + \text{\&c.}$$

$$\text{Cosec } a = \frac{r^2}{a} + \frac{a}{6} + \frac{7a^3}{360r^2} + \frac{31a^5}{15120r^4} + \frac{127a^7}{604800r^6} + \frac{73a^9}{3421440r^8} + \frac{1414477a^{11}}{653837184000r^{10}} \text{ \&c.}$$

56. Also, because vers $a = r - \cos a$, we may obtain, by means of the formulæ for the cosine, the following series for the versed sine; observing that d in the second is equal to the diameter, or $2r$.

$$a = \begin{cases} \sqrt{2r \text{ vers } a}\left(1 + \frac{\text{vers } a}{3.2^2 r} + \frac{3 \text{ vers}^2 a}{4.5.2^3 r^2} + \frac{3.5 \text{ vers}^3 a}{4.6.7.2^4 r^3} + \text{\&c.}\right) \\ \text{Or,} \\ \sqrt{d \text{ vers } a}\left(1 + \frac{\text{vers } a}{2.3\,d}(A) + \frac{3^2 \text{ vers } a}{4.5\,d}(B) + \frac{5^2 \text{ vers } a}{6.7\,d}(C) \text{ \&c.}\right) \end{cases}$$

$$\text{Vers } a =$$

$$\frac{a^2}{2r} - \frac{a^2}{3.4r^2}(A) - \frac{a^2}{5.6r^2}(B) - \frac{a^2}{7.8r^2}(C) - \frac{a^2}{9.10r^2}(D) \text{ \&c.}$$

57. And since covers $a = r - \sin a$, and supv $a = r + \cos a$, we may obtain, in like manner, the following expressions for those lines.

$$\text{Covers } a = r - a + \frac{a^3}{2.3r^2} - \frac{a^5}{2.3.4.5r^4} + \frac{a^7}{2.3.4.5.6.7r^6} - \frac{a^9}{2.3.4.5.6.7.8.9r^8} \text{ \&c.}$$

Supvers $a = 2r -$

$\frac{a^2}{2r^2}$ (A) $- \frac{a^2}{3.4\,r^2}$ (B) $- \frac{a^2}{5.6\,r^2}$ (C) $- \frac{a^2}{7.8\,r^2}$ (D) &c.

58. The sine, cosine, &c. of any arc may also be expressed by means of certain factors, which are found by taking such values of the arc that each side of the equation shall vanish when the sine, or cosine, &c. of the arc, so taken, becomes zero.

Thus, sin $\pm n\pi$ being $= 0$ (art. 13), if sin z $\{= z \times (1 - \frac{z^2}{2.3} + \frac{z^4}{2.3.4.5} -$ &c.$)\}$ be supposed $= z \times z' \times z''$ &c. it is plain that sin z will be 0, whenever z has any of the values that can be taken from the expression $x = \pm n\pi$; and as $1 \mp \frac{z}{n\pi}$ will also be $= 0$, when $\pm n\pi$ is substituted for z, if this be taken as a general factor, independently of the first z, we shall have

$$\text{Sin } z = z \times \left(1 - \frac{z}{\pi}\right) \times \left(1 + \frac{z}{\pi}\right) \times \left(1 - \frac{z}{2\pi}\right) \times \left(1 + \frac{z}{2\pi}\right) \times \left(1 - \frac{z}{3\pi}\right) \times \left(1 + \frac{z}{3\pi}\right) \text{ \&c.}$$

59. In like manner, since cos $\frac{4n \pm 1}{2}\pi = 0$ (art. 13), if cos z $(= 1 - \frac{z^2}{2} + \frac{z^4}{2.3.4}$, &c.$)$ be supposed $= z' \times z'' \times z'''$ &c. it is evident that cos z will be 0, whenever z has any of the values that can be taken from the expression $z = \pm \frac{4n \pm 1}{2}\pi$. And as $1 \mp \frac{2z}{(4n \pm 1)\pi}$ is also $= 0$, when $\pm \frac{4n \pm 1}{2}\pi$ is substituted for z, if this be taken as a general factor, we shall readily obtain the following formula for the cosine of z; viz.

$$\text{Cos } z = \left(1-\frac{2z}{\pi}\right)\times\left(1+\frac{2z}{\pi}\right)\times\left(1-\frac{2z}{3\pi}\right)\times\left(1+\frac{2z}{3\pi}\right)$$
$$\times\left(1-\frac{2z}{5\pi}\right)\times\left(1+\frac{2z}{5\pi}\right)\text{ \&c.}$$

And, if $\frac{m\pi}{2n}$ be substituted for z, in each of the above expressions, we shall have the following formulæ for the sine or cosine of any part of the quadrant or semicircumference.

$$\text{Sin }\frac{m\pi}{2n} = \frac{m\pi}{2n}\left(\frac{2n-m}{2n}\right)\times\left(\frac{2n+m}{2n}\right)\times\left(\frac{4n-m}{4n}\right)\times$$
$$\left(\frac{4n+m}{4n}\right)\times\left(\frac{6n-m}{6n}\right)\times\left(\frac{6n+m}{6n}\right)\text{\&c.}$$

$$\text{Cos }\frac{m\pi}{2n} = \left(\frac{n-m}{n}\right)\times\left(\frac{n+m}{n}\right)\times\left(\frac{3n-m}{3n}\right)\times\left(\frac{3n+m}{3n}\right)$$
$$\times\left(\frac{5n-m}{5n}\right)\times\left(\frac{5n+m}{5n}\right)\text{ \&c.}$$

60. Or, since $\sin\frac{(n-m)\pi}{2n} = \cos\frac{m\pi}{2n}$, and $\cos\frac{(n-m)\pi}{2n} = \sin\frac{m\pi}{2n}$, if $n-m$ be put in the place of m, we shall have (p)

(p) If the first of these expressions, for the sine of $\frac{m\pi}{2n}$, be divided by the second, we shall have $1 = \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{9}{8}\cdot\frac{25}{24}\cdot\frac{49}{48}\text{ \&c.}\right)$ and consequently $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{8}{9}\cdot\frac{24}{25}\cdot\frac{48}{49}\cdot\frac{80}{81}$ &c. $= 2\ (1-\frac{1}{9})\times(1-\frac{1}{25})\times(1-\frac{1}{49})\times(1-\frac{1}{81})$, &c. which is the same as the series given by Wallis in his Arithmetic of Infinites, for the value of $\frac{1}{4}$ of the circumference of the circle.

The first of these formulæ for the sine also gives $\frac{\pi}{2} = \frac{n}{m}\sin\frac{m\pi}{2n}$ $\times\left(\frac{2n}{2n-m}\right)\times\left(\frac{2n}{2n+m}\right)\times\left(\frac{4n}{4n-m}\right)\times\left(\frac{4n}{4n+m}\right)\times\left(\frac{6n}{6n-m}\right)$&c.

$$\text{Cos}\,\frac{m\pi}{2n}=\left(\frac{(n-m)\pi}{2n}\right)\times\left(\frac{n+m}{2n}\right)\times\left(\frac{3n-m}{2n}\right)\times\left(\frac{3n+m}{4n}\right)$$
$$\times\left(\frac{5n-m}{4n}\right)\times\left(\frac{5n+m}{6n}\right)\ \&c.$$

$$\text{Sin}\,\frac{m\pi}{2n}=\frac{m}{n}\left(\frac{2n-m}{n}\right)\times\left(\frac{2n+m}{3n}\right)\times\left(\frac{4n-m}{3n}\right)\times$$
$$\left(\frac{4n+m}{5n}\right)\times\left(\frac{6n-m}{5n}\right)\ \&c.$$

61. Also, if the first of these expressions for the sine be divided by the second for the cosine, and the terms of the series, thus obtained, be afterwards inverted, we shall have

$$\text{Tan}\,\frac{m\pi}{2n}=\frac{m}{n-m}\left(\frac{2n-m}{n+m}\right)\times\left(\frac{2n+m}{3n-m}\right)\times\left(\frac{4n-m}{3n+m}\right)$$
$$\times\left(\frac{4n+m}{5n-m}\right)\ \&c.$$

$$\text{Cot}\,\frac{m\pi}{2n}=\frac{n-m}{m}\left(\frac{n+m}{2n-m}\right)\times\left(\frac{3n-m}{2n+m}\right)\times\left(\frac{3n+m}{4n-m}\right)$$
$$\times\left(\frac{5n-m}{4n+m}\right)\ \&c.$$

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which, on the supposition of $\frac{m}{n}=1$, is the same as the former.

Or, since $\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, we shall have, (by taking $\frac{m}{n}=\frac{1}{2}$) $\frac{\pi}{2}=$ $\frac{\sqrt{2}}{1}\cdot\frac{16}{15}\cdot\frac{64}{63}\cdot\frac{144}{143}\cdot\frac{256}{255}$ &c.

And if this be combined with the series of Wallis it will give $\sqrt{2}=\frac{2^2\times6^2\times10^2\times14^2\times18^2\ \&c.}{1\times3\times5\times7\times9\times11\times13\times15\times17\ \&c.}$ an expression the more remarkable, as it cannot be obtained by any other method.

Also L π = L 4 + L $(1-\frac{1}{9})$ + L $(1-\frac{1}{25})$ + L $(1-\frac{1}{49})$ + L $(1-\frac{1}{81})$ &c. whatever may be the kind of logs., tabular or hyperbolic. And as these logs., when expanded, form series, of which the terms, taken vertically, can be summed, we shall readily obtain hyp. L π = 1.14472985884940017414342, and tab. L π = 0.49714987269413385435126.

62. In like manner, the following expressions for the secants and cosecants may be obtained.

$$\text{Sec}\,\frac{m\pi}{2n} = \frac{n}{n-m}\left(\frac{n}{n+m}\right) \times \left(\frac{3n}{3n-m}\right) \times \left(\frac{3n}{3n+m}\right)$$
$$\times \left(\frac{5n}{5n-m}\right) \times \left(\frac{5n}{5n+m}\right) \text{ \&c.}$$

$$\text{Cosec}\,\frac{m\pi}{2n} = \frac{n}{m}\left(\frac{n}{2n-m}\right) \times \left(\frac{3n}{2n+m}\right) \times \left(\frac{3n}{4n-m}\right)$$
$$\times \left(\frac{5n}{4n+m}\right) \times \left(\frac{5n}{6n-m}\right) \text{ \&c.}$$

63. And if the former expressions for the sine and cosine be combined with the others, we shall have

$$\text{Tan}\,\frac{n\pi}{2n} = \frac{\pi}{2} \times \frac{m}{n-m} \times \frac{1\,(2n-m)}{2\,(n+m)} \times \frac{3\,(2n+m)}{2\,(3n-m)} \times$$
$$\frac{3\,(4n-m)}{4\,(3n+m)} \text{ \&c.}$$

$$\text{Cot}\,\frac{m\pi}{2n} = \frac{\pi}{2} \times \frac{n-m}{n} \times \frac{1\,(n+m)}{2\,(2n-m)} \times \frac{3\,(3n-m)}{2\,(2n+m)}$$
$$\times \frac{3\,(3n+m)}{4\,(4n-m)} \text{ \&c.}$$

$$\text{Sec}\,\frac{m\pi}{2n} = \frac{2}{\pi} \times \frac{n}{n-m} \times \frac{2n}{n+m} \times \frac{2n}{3n-m} \times \frac{4n}{3n+m}$$
$$\times \frac{4n}{5n-m} \text{ \&c.}$$

$$\text{Cosec}\,\frac{m\pi}{2n} = \frac{2}{\pi} \times \frac{n}{m} \times \frac{2n}{2n-m} \times \frac{2n}{2n+m} \times \frac{4n}{4n-m}$$
$$\times \frac{4n}{4n+m} \text{ \&c.}$$

64. Again, if k be put for m, and the sine and cosine of the angle $\frac{k\pi}{2n}$ be found in the same way as the former, we shall have, by dividing the first expressions by the latter, the following formulæ:

$$\frac{\text{Sin}\,\frac{m\pi}{2n}}{\sin\,\frac{k\pi}{2n}} = \frac{m}{k} \times \frac{2n-m}{2n-k} \times \frac{2n+m}{2n+k} \times \frac{4n-m}{4n-k} \times \frac{4n+m}{4n+k} \text{ \&c.}$$

$$\frac{\sin\frac{m\pi}{2n}}{\cos\frac{k\pi}{2n}}=\frac{m}{n-k}\times\frac{2n-m}{n+k}\times\frac{2n+m}{3n-k}\times\frac{4n-m}{3n+k}\times\frac{4n+m}{5n-k}\ \&c.$$

$$\frac{\cos\frac{m\pi}{2n}}{\cos\frac{k\pi}{2n}}=\frac{n-m}{n-k}\times\frac{n+m}{n+k}\times\frac{3n-m}{3n-k}\times\frac{3n+m}{3n+k}\times\frac{5n-m}{5n-k}\ \&c.$$

$$\frac{\cos\frac{m\pi}{2n}}{\sin\frac{k\pi}{2n}}=\frac{n-m}{k}\times\frac{n+m}{2n-k}\times\frac{3n-m}{2n+k}\times\frac{3n+m}{4n-k}\times\frac{5n-m}{4n+k}\ \&c.$$

Whence, taking an angle $\frac{k\pi}{2n}$, of which the sine and cosine are given, we can readily find the sine and cosine of any other angle $\frac{m\pi}{2n}$.

65. From these and the former expressions, the natural and logarithmic sines, cosines, &c. may be obtained much more readily than by the methods employed by the first calculators of our present tables. For since

$$\sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7}\ \&c.$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6}\ \&c.$$

if $\frac{m}{n}\left(\frac{\pi}{2}\right)$ be substituted for x, we shall have

$$\sin\frac{m}{n}\left(\frac{\pi}{2}\right)=\frac{m}{n}\cdot\frac{\pi}{2}-\frac{1}{2.3}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)^3+\frac{1}{2.3.4.5}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)^5-\&c.$$

$$\cos\frac{m}{n}\left(\frac{\pi}{2}\right)=1-\tfrac{1}{2}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)^2+\frac{1}{2.3.4}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)^4-\&c.$$

And by taking $\pi = 3.1415926535897932$, and calculating the coefficients to 16 decimals, the following formulæ will be readily obtained (*q*):

(*q*) See the *Analysis Infinitorum* of Euler, or the French translation of that work, with notes, by Labey: where these and various other series, of a similar kind, are investigated.

$\text{Sin}\,\frac{m}{n}\,.\,90^\circ =$	$\text{Cos}\,\frac{m}{n}\,.\,90^\circ =$
$1.5707963267948966\frac{m}{n}$	1.0000000000000000
$-0.6459640975062463\frac{m^3}{n^3}$	$-1.2337005501361698\frac{m^2}{n^2}$
$+0.0796926262461670\frac{m^5}{n^5}$	$+0.2536695079010480\frac{m^4}{n^4}$
$-0.0046817541353187\frac{m^7}{n^7}$	$-0.0208634807633530\frac{m^6}{n^6}$
$+0.0001604411847874\frac{m^9}{n^9}$	$+0.0009192602748394\frac{m^8}{n^8}$
$-0.0000035988432352\frac{m^{11}}{n^{11}}$	$-0.0000252020423731\frac{m^{10}}{n^{10}}$
$+0.0000000569217292\frac{m^{13}}{n^{13}}$	$+0.0000004710874779\frac{m^{12}}{n^{12}}$
$-0.0000000006688035\frac{m^{15}}{n^{15}}$	$-0.0000000063866031\frac{m^{14}}{n^{14}}$
$+0.0000000000060669\frac{m^{17}}{n^{17}}$	$+0.0000000000656596\frac{m^{16}}{n^{16}}$
$-0.0000000000000438\frac{m^{19}}{n^{19}}$	$-0.0000000000005294\frac{m^{18}}{n^{18}}$
$+0.0000000000000003\frac{m^{21}}{n^{21}}$	$+0.0000000000000034\frac{m^{20}}{n^{20}}$

66. Also, because $\sin\frac{m\pi}{2n} = \frac{m\pi}{2n}\left(\frac{2n-m}{2n}\right) \times \left(\frac{2n+m}{2n}\right) \times \left(\frac{4n-m}{4n}\right) \times \left(\frac{4n+m}{4n}\right)$ &c. and $\cos\frac{m\pi}{2n} = \frac{n-m}{n}\left(\frac{n+m}{n}\right) \times \left(\frac{3n-m}{3n}\right) \times \left(\frac{3n+m}{3n}\right)$ &c. if these factors be multiplied two by two, we shall have, by taking their logarithms,

$$\text{L}\sin\frac{m\pi}{2n} = \text{L}\frac{m\pi}{2n} + \text{L}\left(1-\frac{m^2}{4n^2}\right) + \text{L}\left(1-\frac{m^2}{16n^2}\right) + \text{L}\left(1-\frac{m^2}{36n}\right) + \text{L}\left(1-\frac{m^2}{64n^2}\right) \text{ \&c.}$$

$$\text{L}\cos\frac{m\pi}{2n} = \text{L}\left(1-\frac{m^2}{n^2}\right) + \text{L}\left(1-\frac{m^2}{9n^2}\right) + \text{L}\left(1-\frac{m^2}{25n^2}\right) + \text{L}\left(1-\frac{m^2}{49n^2}\right) + \text{L}\left(1-\frac{m^2}{81n^2}\right) \text{ \&c.}$$

67. And by reserving the logarithms of a few of the first terms, in order to render the series more rapid, and calculating the coefficients of the expanded terms of the others, the following formulæ for the tabular log sines and cosines of any arc may be readily obtained.

$\text{L} \sin \frac{m}{n} 90^\circ =$	$\text{Cos} \frac{m}{n} 90^\circ =$
$\text{L}m + \text{L}(2n-m) + \text{L}(2n+m) - 3\text{L}n$	$\text{L}(n-m) + \text{L}(n+m) - 2\text{L}n$
$+9.5940598857021903$	$+10.0000000000000000$
$-0.0700228266059019\frac{m^2}{n^2}$	$-\ 0.1014948593418928\frac{m^2}{n^2}$
$-0.0011172664416618\frac{m^4}{n^4}$	$-\ 0.0031872940654511\frac{m^4}{n^4}$
$-0.0000392291464539\frac{m^6}{n^6}$	$-\ 0.0002094858000174\frac{m^6}{n^6}$
$-0.0000017292707984\frac{m^8}{n^8}$	$-\ 0.0000168483485983\frac{m^8}{n^8}$
$-0.0000000843629863\frac{m^{10}}{n^{10}}$	$-\ 0.0000014801939869\frac{m^{10}}{n^{10}}$
$-0.0000000043487155\frac{m^{12}}{n^{12}}$	$-\ 0.0000001365022722\frac{m^{12}}{n^{12}}$
$-0.0000000002319312\frac{m^{14}}{n^{14}}$	$-\ 0.0000000129817147\frac{m^{14}}{n^{14}}$
$-0.0000000000126591\frac{m^{16}}{n^{16}}$	$-\ 0.0000000012614711\frac{m^{16}}{n^{16}}$
$-0.0000000000007027\frac{m^{18}}{n^{18}}$	$-\ 0.0000000001245671\frac{m^{18}}{n^{18}}$
$-0.0000000000000395\frac{m^{20}}{n^{20}}$	$-\ 0.0000000000124559\frac{m^{20}}{n^{20}}$
$-0.0000000000000022\frac{m^{22}}{n^{22}}$	$-\ 0.0000000000012581\frac{m^{22}}{n^{22}}$

68. Hence, as the sines and cosines of arcs, from zero to 45°, comprehend the sines and cosines of arcs from 45° to 90°, $\frac{m}{n}$ in these formulæ may always be taken less than $\frac{1}{2}$; and as the series are thus rendered very convergent, it will only be necessary, in many

cases, to calculate a few of their terms, to obtain the sine or cosine of the arc required.

69. Several other formulæ for expressing the sine, cosine, &c. in functions of the arc, may also be obtained by means of certain imaginary factors, which have been found of considerable use in physical astronomy, and various other branches of the modern analysis.

Thus, radius being supposed $= 1$, we shall have $\cos^2 x + \sin^2 x = 1$, of which the first member of the equation is the product of the two imaginary factors $\cos x + \sin x \sqrt{-1}$ and $\cos x - \sin x \sqrt{-1}$. And if any two similar factors $\cos x \pm \sin x \sqrt{-1}$ and $\cos y \pm \sin y \sqrt{-1}$ be multiplied together, their product will be $= \cos x \cos y - \sin x \sin y \pm (\sin x \cos y + \sin y \cos x) \sqrt{-1} = \cos(x + y) \pm \sin(x + y) \sqrt{-1}$.

In like manner, $(\cos x \pm \sin x \sqrt{-1}) \times (\cos y \pm \sin y \sqrt{-1}) \times (\cos z \pm \sin z \sqrt{-1}) = \cos(x+y+z) \pm \sin(x+y+z) \sqrt{-1}$, &c. each of which are like the simple factors, and are produced in a similar way with logarithms, by barely adding the arcs.

And if the arcs x, y, z, &c. be supposed equal to each other, we shall have $(\cos x \pm \sin x \sqrt{-1})^2 = \cos 2x \pm \sin 2x \sqrt{-1}$, and for the three factors $(\cos x \pm \sin x \sqrt{-1})^3 = \cos 3x \pm \sin 3x \sqrt{-1}$. Consequently, in general,

$$(\operatorname{Cos} x \pm \sin x \sqrt{-1})^n = \cos nx \pm \sin nx \sqrt{-1}.$$

Hence, by transposition and division, we shall obtain the two following equations for the sine and cosine of any multiple of the arc x, in terms of the arc.

$$\text{Cos } n\,x = \frac{(\cos x + \sin x\sqrt{-1})^n + (\cos x - \sin x\sqrt{-1})^n}{2}$$

$$\text{Sin } n\,x = \frac{(\cos x + \sin x\sqrt{-1})^n - (\cos x - \sin x\sqrt{-1})^n}{2\sqrt{-1}}$$

70. The sine, cosine, &c. of any arc, or multiple of that arc, may also be readily derived from the well-known exponential expression $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4}$, &c. where e is the number of which the hyperbolic logarithm is 1.

For, if in this formula, $x\sqrt{-1}$ and $-x\sqrt{-1}$ be successively substituted for z, we shall have

$$e^{x\sqrt{-1}} = 1 + \frac{x\sqrt{-1}}{1} - \frac{x^2}{2} - \frac{x^3\sqrt{-1}}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5\sqrt{-1}}{2.3.4.5} - \text{\&c.}$$

$$e^{-x\sqrt{-1}} = 1 - \frac{x\sqrt{-1}}{1} - \frac{x^2}{2} + \frac{x^3\sqrt{-1}}{2.3} + \frac{x^4}{2.3.4} - \frac{x^5\sqrt{-1}}{2.3.4.5} - \text{\&c.}$$

And if these be added to and subtracted from each other, the results, after being divided by 2, and $2\sqrt{-1}$, will give

$$\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \text{\&c.}$$

$$\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}} = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \text{\&c.}$$

Of which series the second members are the known values, as before found, of the cosine of x and sine of x. Whence

$$\text{Cos } x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2};\ \sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$$

$$\text{Tan } x = \frac{1}{\sqrt{-1}} \times \left(\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}\right); \cot x = \sqrt{-1}\left(\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{e^{x\sqrt{\ }1} - e^{-x\sqrt{-1}}}\right)$$

Or, if each of the terms of the numerator and denominator of the second members of the two last equations be multiplied by $e^{x\sqrt{-1}}$, they will become

$$\text{Tan } x = \frac{1}{\sqrt{-1}}\left(\frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1}\right); \cot x = \sqrt{-1}\left(\frac{e^{2x\sqrt{-1}} + 1}{e^{2x\sqrt{-1}} - 1}\right)$$

And, if $m\,x$ be substituted for x, in each of these formulæ, we shall obtain the sine, cosine, &c. of any multiple of those arcs: thus, $\text{Cos } m\,x =$

$$\frac{e^{m\,x\sqrt{-1}} + e^{-m\,x\sqrt{-1}}}{2}; \ \sin m\,x = \frac{e^{m\,x\sqrt{-1}} - e^{-m\,x\sqrt{-1}}}{2\sqrt{-1}}$$

$$\text{Tan } m\,x = \frac{1}{\sqrt{-1}} \times \left(\frac{e^{2m\,x\sqrt{-1}} - 1}{e^{2m\,x\sqrt{-1}} + 1}\right); \ \cot m\,x = \sqrt{-1}\left(\frac{e^{2m\,x\sqrt{\ }1} + 1}{e^{2m\,x\sqrt{-1}} - 1}\right)$$

71. The same formulæ also give $e^{x\sqrt{-1}} = \cos x + \sin x\sqrt{-1}$, and $e^{-x\sqrt{-1}} = \cos x - \sin x\sqrt{-1}$; whence, by division, we shall have $e^{2x\sqrt{-1}} = \frac{\cos x + \sin x\sqrt{-1}}{\cos x - \sin x\sqrt{-1}} = \frac{1 + \tan x\sqrt{-1}}{1 - \tan x\sqrt{-1}}$; and by taking the logarithms of each member, $2x\sqrt{-1} = \log\left(\frac{1 + \tan x\sqrt{-1}}{1 - \tan x\sqrt{-1}}\right)$ But $\log\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7 \text{ \&c.}\right)$; therefore, putting $\tan x\sqrt{-1}$ in the place of z, and dividing each member of the equation by $4\sqrt{-1}$, we shall obtain

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x \text{ \&c.}$$

Which is the known series for the value of any arc in terms of its tangent.

72. It may here also be shown, that any trigonometrical expression, of the form $\tan x = \frac{m \sin z}{1 \pm m \cos z}$ can be converted into a series of the various multiples of the sine of z, taken progressively.

For, if in this equation, there be put instead of the tangent of x and the sine and cosine of z, their values in exponentials (art. 70), we shall obtain

$$\frac{e^{2x\sqrt{-1}}-1}{e^{2x\sqrt{-1}}+1} = \frac{m\left(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}\right)}{2 \pm m\left(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}\right)}$$

And, consequently, by reduction,

$$e^{2x\sqrt{-1}} = \frac{1-me^{x\sqrt{-1}}}{1+me^{-x\sqrt{-1}}} \text{ or } \frac{1+me^{-x\sqrt{-1}}}{1-me^{x\sqrt{-1}}}$$

Whence, by taking the logarithm of each member of the equation, and converting the second into a series, according to the form $\log(1 \pm z) = z - \frac{1}{2}z^2 \pm \frac{1}{3}z^3 - \frac{1}{4}z^4$ &c. we shall have $2x\sqrt{-1} =$

$$\begin{cases} me^{x\sqrt{-1}} - \frac{m^2}{2}e^{2x\sqrt{-1}} + \frac{m^3}{3}e^{3x\sqrt{-1}} - \frac{m^4}{4}e^{4x\sqrt{-1}} \\ -me^{-x\sqrt{-1}} + \frac{m^2}{2}e^{-2x\sqrt{-1}} - \frac{m^3}{3}e^{-3x\sqrt{-1}} + \frac{m^4}{4}e^{4x\sqrt{-1}}. \end{cases}$$

But $e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sin x\sqrt{-1}$, $e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}} = 2\sin 2x\sqrt{-1}$ &c. (art. 70); whence, dividing each side of the equation by $2\sqrt{-1}$, we shall have

$$x = m\sin x - \frac{m^2}{2}\sin 2x + \frac{m^3}{3}\sin 3x - \frac{m^4}{4}\sin 4x + \&c.$$

Or,

$$x = m\sin x - \frac{m^2}{2}\sin 2x - \frac{m^3}{3}\sin 3x - \frac{m^4}{4}\sin 4x - \&c.$$

The former of which series answers to the case $\tan x = \frac{m \sin z}{1 + m \cos z}$, and the latter to $\tan x = \frac{m \sin z}{1 - m \cos z}$ (r).

73. Several other expressions for the sine, cosine, &c. of multiple arcs, may be derived from one or other of the preceding formulæ; the most curious and useful of which are the following:

$\text{Sin } z = 1 \sin z$

$\text{Sin } 2z = 2 \sin z \sin \left(\frac{\pi}{2} - z\right)$

$\text{Sin } 3z = 4 \sin z \sin \left(\frac{\pi}{3} - z\right) \sin \left(\frac{\pi}{3} + z\right)$

$\text{Sin } 4z = 8 \sin z \sin \left(\frac{\pi}{4} - z\right) \sin \left(\frac{\pi}{4} + z\right) \sin \left(\frac{2\pi}{4} - z\right)$

$\text{Sin } 5z = 16 \sin z \sin \left(\frac{\pi}{5} - z\right) \sin \left(\frac{\pi}{5} + z\right) \sin \left(\frac{2\pi}{5} - z\right)$
$\sin \left(\frac{2\pi}{5} + z\right)$ &c.

Or, generally,

$\text{Sin } nz = 2^{n-1} \sin z \sin \left(\frac{\pi}{n} - z\right) \sin \left(\frac{\pi}{n} + z\right) \sin \left(\frac{2\pi}{n} - z\right) \sin \left(\frac{2\pi}{n} + z\right) \sin \left(\frac{3\pi}{n} - z\right) \sin \left(\frac{3\pi}{n} + z\right)$ &c.

Where the series must be continued to as many factors as there are units in n.

74. $\text{Cos } z = \sin \left(\frac{\pi}{2} - z\right)$

$\text{Cos } 2z = 2 \sin \left(\frac{\pi}{4} - z\right) \sin \left(\frac{\pi}{4} + z\right)$

(r) The simple series $z = \sin z - \frac{1}{2} \sin 2z + \frac{1}{3} \sin 3z - \frac{1}{4} \sin 4z$, &c. of which that given above is a more general form, was first discovered by Euler; as was, also, the series $x = \sin x \sec \frac{1}{2} x \sec \frac{1}{4} x \sec \frac{1}{8} x$, &c.

For the algebraical solutions of several of the cases of spherical triangles, according to this form, see Tables Trigonométriques de Borda; where the following theorem is likewise given for the decimal division of the circle, $x = \frac{m \sin z}{\sin 1''} \mp \frac{\frac{1}{2} m^2 \sin 2z}{\sin 1''} + \frac{\frac{1}{3} m^3 \sin 3z}{\sin 1''} \mp$ &c.

$$\text{Cos } 3z = 4 \sin\left(\frac{\pi}{6} - z\right) \sin\left(\frac{\pi}{6} + z\right) \sin\left(\frac{3\pi}{6} - z\right)$$

$$\text{Cos } 4z = 8 \sin\left(\frac{\pi}{8} - x\right) \sin\left(\frac{\pi}{8} + z\right) \sin\left(\frac{3\pi}{8} - z\right) \sin\left(\frac{3\pi}{8} + z\right)$$

$$\text{Cos } 5z = 16 \sin\left(\frac{\pi}{10} - z\right) \sin\left(\frac{\pi}{10} + z\right) \sin\left(\frac{3\pi}{10} - z\right) \sin\left(\frac{3\pi}{10} + z\right) \sin\left(\frac{5\pi}{10} - z\right)$$

&c. Or, generally,

$$\text{Cos } nz = 2^{n-1} \sin\left(\frac{\pi}{2n} - z\right) \sin\left(\frac{\pi}{2n} + z\right) \sin\left(\frac{3\pi}{2n} - z\right) \sin\left(\frac{3\pi}{2n} + z\right) \sin\left(\frac{5\pi}{2n} - z\right) \sin\left(\frac{5\pi}{2n} + z\right) \text{ \&c. to } n \text{ factors.}$$

75. $\text{Cos } z = \cos z$

$$\text{Cos } 2z = 2 \cos\left(\frac{\pi}{4} + z\right) \cos\left(\frac{\pi}{4} - z\right)$$

$$\text{Cos } 3z = 4 \cos\left(\frac{2\pi}{6} + z\right) \cos\left(\frac{2\pi}{6} - z\right) \cos z$$

$$\text{Cos } 4z = 8 \cos\left(\frac{3\pi}{8} + z\right) \cos\left(\frac{3\pi}{8} - z\right) \cos\left(\frac{\pi}{8} + z\right) \cos\left(\frac{\pi}{8} - z\right)$$

$$\text{Cos } 5z = 16 \cos\left(\frac{4\pi}{10} + z\right) \cos\left(\frac{4\pi}{10} - z\right) \cos\left(\frac{2\pi}{10} + z\right) \cos\left(\frac{2\pi}{10} - z\right) \cos z.$$

&c. Or, generally,

$$\text{Cos } nz = 2^{n-1} \cos\left(\frac{n-1}{2n}\pi + z\right) \cos\left(\frac{n-1}{2n}\pi - z\right) \cos\left(\frac{n-3}{2n}\pi + z\right) \cos\left(\frac{n-3}{2n}\pi - z\right) \cos\left(\frac{n-5}{2n}\pi + z\right) \cos\left(\frac{n-5}{2n}\pi - z\right) \cos\left(\frac{n-7}{2n}\pi + z\right) \text{ \&c. to } n \text{ factors.}$$

76. $\text{Sec } z = \sec z$

$$3 \text{ Sec } 3z = \sec\left(\frac{\pi}{3} + z\right) + \sec\left(\frac{\pi}{3} - z\right) - \sec\left(\frac{z}{3}\right.$$

$5 \operatorname{Sec} 5z = \sec\left(\frac{2\pi}{5} + z\right) + \sec\left(\frac{2\pi}{5} - z\right) - \sec\left(\frac{\pi}{5} + z\right)$
$- \sec\left(\frac{\pi}{5} - z\right) + \sec z$

$7 \operatorname{Sec} 7z = \sec\left(\frac{3\pi}{7} + z\right) + \sec\left(\frac{3\pi}{7} - z\right) - \sec\left(\frac{2\pi}{7} + z\right)$
$- \sec\left(\frac{2\pi}{7} - z\right) + \sec\left(\frac{\pi}{7} + z\right) + \sec\left(\frac{\pi}{7} - z\right) - \sec z$

&c.

Or, generally, putting $n = 2m + 1$

$n \sec nz = \sec\left(\frac{m}{n}\pi + z\right) + \sec\left(\frac{m}{n}\pi - z\right) - \sec$
$\left(\frac{m-1}{n}\pi + z\right) - \sec\left(\frac{m-1}{n}\pi - z\right) + \sec\left(\frac{m-2}{n}\pi + z\right)$
$+ \sec\left(\frac{m-2}{n}\pi - z\right) - - - - - - - \pm \sec z.$

77. $\operatorname{Cosec} z = \operatorname{cosec} z$

$3 \operatorname{Cosec} 3z = \operatorname{cosec} z + \operatorname{cosec}\left(\frac{\pi}{3} - z\right) - \operatorname{cosec}\left(\frac{\pi}{3} + z\right)$

$5 \operatorname{Cosec} 5z = \operatorname{cosec} z + \operatorname{cosec}\left(\frac{\pi}{5} - z\right) - \operatorname{cosec}\left(\frac{\pi}{5} - z\right)$
$- \operatorname{cosec}\left(\frac{2\pi}{5} - z\right) + \operatorname{cosec}\left(\frac{2\pi}{5} + z\right).$

&c. Or, generally, putting $n = 2m + 1$

$n \operatorname{cosec} nz = \operatorname{cosec} z + \operatorname{cosec}\left(\frac{\pi}{n} - z\right) - \cos\left(\frac{\pi}{n} + z\right)$
$- \operatorname{cosec}\left(\frac{2\pi}{n} - z\right) + \operatorname{cosec}\left(\frac{2\pi}{n} + z\right) + \operatorname{cosec}\left(\frac{3\pi}{n} - z\right)$
$- \operatorname{cosec}\left(\frac{3\pi}{n} + z\right) - - - - \mp \operatorname{cosec}\left(\frac{m\pi}{n} - z\right) \pm \operatorname{cosec}$
$\left(\frac{m\pi}{n} + z\right)$

Where the upper signs take place when m is an even number, and the lower when it is an odd number.

78. $\operatorname{Tan} z = \tan z$

$3 \operatorname{Tan} 3z = \tan z + \tan\left(\frac{\pi}{3} + z\right) + \tan\left(\frac{2\pi}{3} + z\right)$

$5 \tan 5z = \tan z + \tan\left(\frac{\pi}{5}+z\right) + \tan\left(\frac{2\pi}{5}+z\right) + \tan\left(\frac{3\pi}{5}+z\right) + \tan\left(\frac{4\pi}{5}+z\right)$

&c.

Or, since $\tan v = -\tan(\pi - v)$, we shall have

$\operatorname{Tan} z = \tan z$

$3 \tan 3z = \tan z - \tan\left(\frac{\pi}{3} - z\right) + \tan\left(\frac{\pi}{3} + z\right)$

$5 \tan 5z = \tan z - \tan\left(\frac{\pi}{5} - z\right) + \tan\left(\frac{\pi}{5} + z\right) - \tan\left(\frac{2\pi}{5} - z\right) + \tan\left(\frac{2\pi}{5} + z\right)$

&c.

Or, generally, putting $n = 2m + 1$

$n \tan nz = \tan z - \tan\left(\frac{\pi}{n} - z\right) + \tan\left(\frac{\pi}{n} + z\right) - \tan\left(\frac{2\pi}{n} - z\right) + \tan\left(\frac{2\pi}{n} + z\right) - \tan\left(\frac{3\pi}{n} - z\right) + \cdots - \tan\left(\frac{m\pi}{n} - z\right) + \tan\left(\frac{m\pi}{n} + z\right)$.

Also,

$\operatorname{Tan} z = \tan z$

$\operatorname{Tan} 3z = \tan z \tan\left(\frac{\pi}{3} - z\right) \tan\left(\frac{\pi}{3} + z\right)$

$\operatorname{Tan} 5z = \tan z \tan\left(\frac{\pi}{5} - z\right) \tan\left(\frac{\pi}{5} + z\right) \tan\left(\frac{2\pi}{5} - z\right) \tan\left(\frac{2\pi}{5} + z\right)$.

Or, generally, putting $n = 2m + 1$, as before,

$\operatorname{Tan} nz = \tan z \tan\left(\frac{\pi}{n} - z\right) \tan\left(\frac{\pi}{n} + z\right) \tan\left(\frac{2\pi}{n} - z\right) \tan\left(\frac{2\pi}{n} + z\right) \tan\left(\frac{3\pi}{n} - z\right) \cdots\cdots \times \tan\left(\frac{m\pi}{n} - z\right) \tan\left(\frac{m\pi}{n} + z\right)$.

79. $\operatorname{Cot} z = \cot z$

$2 \operatorname{Cot} 2z = \cot z + \cot\left(\frac{\pi}{2} + z\right)$

$3\,\mathrm{Cot}\,3z = \cot z + \cot\left(\frac{\pi}{3} + z\right) + \cot\left(\frac{2\pi}{3} + z\right)$

$4\,\mathrm{Cot}\,4z = \cot z + \cot\left(\frac{\pi}{4} + z\right) + \cot\left(\frac{2\pi}{4} + z\right) + \cot$
$\left(\frac{3\pi}{4} + z\right)$

$5\,\mathrm{Cot}\,5z = \cot z + \cot\left(\frac{\pi}{5} + z\right) + \cot\left(\frac{2\pi}{5} + z\right) + \cot$
$\left(\frac{3\pi}{5} + z\right) + \cot\left(\frac{4\pi}{5} + z\right).$

&c.

Or, since $\cot v = -\cot(\pi - v)$, we shall have

$\mathrm{Cot}\,z = \cot z$

$2\,\mathrm{Cot}\,2z = \cot z - \cot\left(\frac{\pi}{2} - z\right)$

$3\,\mathrm{Cot}\,3z = \cot z - \cot\left(\frac{\pi}{3} - z\right) + \cot\left(\frac{\pi}{3} + z\right)$

$4\,\mathrm{Cot}\,4z = \cot z - \cot\left(\frac{\pi}{4} - z\right) + \cot\left(\frac{\pi}{4} + z\right) - \cot$
$\left(\frac{2\pi}{4} - z\right)$

$5\,\mathrm{Cot}\,5z = \cot z - \cot\left(\frac{\pi}{5} - z\right) + \cot\left(\frac{\pi}{5} + z\right) - \cot$
$\left(\frac{2\pi}{5} - z\right) + \cot\left(\frac{2\pi}{5} + z\right).$

&c. Or, generally,

$n \cot nz = \cot z - \cot\left(\frac{\pi}{n} - z\right) + \cot\left(\frac{\pi}{n} + z\right) -$
$\cot\left(\frac{2\pi}{n} - z\right) + \cot\left(\frac{2\pi}{n} + z\right) - \cot\left(\frac{3\pi}{n} - z\right) + \cot\left(\frac{3\pi}{n}\right.$
$\left.+ z\right) -$ &c. to n terms.

Also,

$-2\cot 2z = \tan z + \tan\left(\frac{\pi}{2} + z\right)$

$-4\cot 4z = \tan z + \tan\left(\frac{\pi}{4} + z\right) + \tan\left(\frac{2\pi}{4} + z\right) +$
$\tan\left(\frac{3\pi}{4} + z\right)$

$-6\cot 6z = \tan z + \tan\left(\frac{\pi}{6} + z\right) + \tan\left(\frac{2\pi}{6} + z\right) +$
$\tan\left(\frac{3\pi}{6} + z\right) + \tan\left(\frac{4\pi}{6} + z\right) + \tan\left(\frac{5\pi}{6} + z\right)$

&c.

Or,

$$2 \operatorname{Cot} 2z = -\tan x + \tan\left(\frac{\pi}{2} - z\right)$$

$$4 \operatorname{Cot} 4z = -\tan z + \tan\left(\frac{\pi}{4} - z\right) - \tan\left(\frac{\pi}{4} + z\right) +$$
$$\tan\left(\frac{2\pi}{4} - z\right)$$

$$6 \operatorname{Cot} 6z = -\tan z + \tan\left(\frac{\pi}{6} - z\right) - \tan\left(\frac{\pi}{6} + z\right) +$$
$$\tan\left(\frac{2\pi}{6} - z\right) - \tan\left(\frac{2\pi}{6} + z\right) + \tan\left(\frac{3\pi}{6} - z\right)$$

&c. Or, generally,

$$n \cot nz = -\tan z + \tan\left(\frac{\pi}{n} - z\right) - \tan\left(\frac{\pi}{n} + z\right) +$$
$$\tan\left(\frac{2\pi}{n} - z\right) - \tan\left(\frac{2\pi}{n} + z\right) + \cdots + \tan\left(\frac{m\pi}{n} - z\right).$$

80. The sum of the sines of any number of arcs in arithmetical progression, may be also exhibited as follows:

$$\operatorname{Sin} a + \sin(a+b) + \sin(a+2b) + \sin(a+3b)$$
$$+ \sin(a+4b) \text{ \&c. ad infinitum} = \frac{\cos\left(a - \frac{1}{2}b\right)}{2\sin\frac{1}{2}b}.$$

Also,

$$\operatorname{Sin} a + \sin(a+b) + \sin(a+2b) + \sin(a+3b) + \sin$$
$$(a+4b) + \cdots \sin(a+nb) = \frac{\sin\left(a+\frac{1}{2}nb\right)\sin\frac{1}{2}(n+1)b}{\sin\frac{1}{2}b}.$$

81. And the sum of the cosines of any number of arcs, similarly taken, is as below:

$$\operatorname{Cos} a + \cos(a+b) + \cos(a+2b) + \cos(a+3b)$$
$$+ \cos(a+4b), \text{ \&c. ad infinitum} = -\frac{\sin\left(a-\frac{1}{2}b\right)}{2\sin\frac{1}{2}b}.$$

Also,

$$\operatorname{Cos} a + \cos(a+b) + \cos(a+2b) + \cos(a+3b) + \cos$$
$$(a+4b)\text{\&c.} \cdots + \cos(a+nb) = \frac{\cos\left(a+\frac{1}{2}nb\right)\sin\frac{1}{2}(n+1)b}{\sin\frac{1}{2}b}.$$

82. As some of the common logarithmic formulæ, for numbers, are often required in the investigations of trigonometrical expressions, I shall here subjoin such of them as are the most useful and necessary, for the sake of reference. Thus,

$$\text{Log}\,(1+p)=\frac{1}{M}\left(p-\tfrac{1}{2}p^2+\tfrac{1}{3}p^3-\tfrac{1}{4}p^4+\tfrac{1}{5}p^5-\&c.\right)$$

$$\text{Log}\,\frac{1}{1-p}=\frac{1}{M}\left(p+\tfrac{1}{2}p^2+\tfrac{1}{3}p^3+\tfrac{1}{4}p^4+\tfrac{1}{5}p^5+\tfrac{1}{6}p^6\ \&c.\right)$$

$$\text{Log}\,\frac{1+p}{1-p}=\frac{2}{M}\left(p+\tfrac{1}{3}p^3+\tfrac{1}{5}p^5+\tfrac{1}{7}p^7+\tfrac{1}{9}p^9+\tfrac{1}{11}p^{11}\ \&c.\right)$$

Or, $\text{Log}\,a=$

$$\frac{1}{M}\left\{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4\ \&c.\right\}$$

$\text{Log}\,a=$

$$\frac{1}{M}\left\{\frac{a-1}{a}+\tfrac{1}{2}\left(\frac{a-1}{a}\right)^2+\tfrac{1}{3}\left(\frac{a-1}{a}\right)^3+\tfrac{1}{4}\left(\frac{a-1}{a}\right)^4+\&c.\right\}$$

$\text{Log}\,a=$

$$\frac{2}{M}\left\{\frac{a-1}{a+1}+\tfrac{1}{3}\left(\frac{a-1}{a+1}\right)^3+\tfrac{1}{5}\left(\frac{a-1}{a+1}\right)^5+\tfrac{1}{7}\left(\frac{a-1}{a+1}\right)^7+\&c.\right\}$$

Also, $\text{Log}\,\frac{a}{b}=$

$$\frac{1}{M}\left\{\frac{a\sim b}{b}-\tfrac{1}{2}\left(\frac{a\sim b}{b}\right)^2+\tfrac{1}{3}\left(\frac{a\sim b}{b}\right)^3-\tfrac{1}{4}\left(\frac{a\sim b}{b}\right)^4+\&c.\right\}$$

$\text{Log}\,\frac{a}{b}=$

$$\frac{1}{M}\left\{\frac{a\sim b}{a}+\tfrac{1}{2}\left(\frac{a\sim b}{a}\right)^2+\tfrac{1}{3}\left(\frac{a\sim b}{a}\right)^3+\tfrac{1}{4}\left(\frac{a\sim b}{a}\right)^4+\&c.\right\}$$

$\text{Log}\,\frac{a}{b}=$

$$\frac{2}{M}\left\{\frac{a\sim b}{a+b}+\tfrac{1}{3}\left(\frac{a\sim b}{a+b}\right)^3+\tfrac{1}{5}\left(\frac{a\sim b}{a+b}\right)^5+\tfrac{1}{7}\left(\frac{a\sim b}{a+b}\right)^7+\&c.\right\}$$

Or, $\text{Log}\,a=$

$$\text{Log}\,(a-1)+\frac{1}{M}\left\{\frac{1}{a}+\frac{1}{2a^2}+\frac{1}{3a^3}+\frac{1}{4a^4}+\frac{1}{5a^5}\ \&c.\right\}$$

$$\text{Log } a = \log (a-1) + \frac{1}{\text{M}}\left\{\frac{1}{a-1} - \frac{1}{2(a-1)^2} + \frac{1}{3(a-1)^3} - \frac{1}{4(a-1)^4} + \&c.\right\}$$

$$\text{Log } a = \log (a-1) + \frac{2}{\text{M}}\left\{\frac{1}{2a-1} + \frac{1}{3(2a-1)^3} + \frac{1}{5(2a-1)^5} + \frac{1}{7(2a-1)^7} + \&c.\right\}$$

Where M = 1 for hyperbolic logarithms, or = 2.302585093 for the common tabular logarithms; which number is the hyperbolic logarithm of 10, or what is usually called the modulus of the system. And if its reciprocal be used, it becomes $\frac{1}{\text{M}} = .434294482$ for a multiplier.

83. To these formulæ may also be added the following, which will be found useful upon particular occasions (*t*).

$$\text{Log } a = \frac{1}{\text{M}}\left\{\left(a-a^{-1}\right) - \tfrac{1}{2}\left(a^2-a^{-2}\right) + \tfrac{1}{3}\left(a^3-a^{-3}\right) - \tfrac{1}{4}\left(a^4-a^{-4}\right) \&c.\right\}$$

(*t*) For an investigation of the doctrine of logarithms, from the most simple algebraical principles, see article Logarithms, given by the Author, in the Supplement to Hutton's Mathematical Dictionary, or Lagrange, Théorie des Fonctions Analytiques.

To the logarithmic formulæ, given above, may also be added those of Borda and Haros, cited by Lacroix, in his Leçons d'Algèbre, which are as follows:

$$\text{Log } (n+2) = \log (n-2) + 2 \log (n+1) - 2 \log (n-1) + \frac{2}{\text{M}}\left\{\frac{2}{n^3-3n} + \tfrac{1}{3}\left(\frac{2}{n^3-3n}\right)^3 + \tfrac{1}{5}\left(\frac{2}{n^3-3n}\right)^5 \&c.\right\}$$

$$\text{Log } (n+5) = \log (n-3) + \log (n+3) + \log (n-4) + \log (n+4) - \log (n-5) - 2 \log n - \frac{2}{\text{M}}\left\{\frac{72}{n^4-25n^2+72} + \tfrac{1}{3}\left(\frac{72}{n^4-25n^2+72}\right)^3 + \tfrac{1}{5}\left(\frac{72}{n^4-25n^2+72}\right)^5 + \&c.\right\}$$

$$\text{Log}\,(a+z) =$$
$$\text{Log}\,a + \frac{1}{\text{M}}\left\{\frac{z}{a} - \tfrac{1}{2}\left(\frac{z}{a}\right)^2 + \tfrac{1}{3}\left(\frac{z}{a}\right)^3 - \tfrac{1}{4}\left(\frac{z}{a}\right)^4 + \&\text{c.}\right\}$$
$$\text{Log}\,(a-z) =$$
$$\text{Log}\,a - \frac{1}{\text{M}}\left\{\frac{z}{a} + \tfrac{1}{2}\left(\frac{z}{a}\right)^2 + \tfrac{1}{3}\left(\frac{z}{a}\right)^3 + \tfrac{1}{4}\left(\frac{z}{a}\right)^4 + \&\text{c.}\right\}$$
$$\text{Log}\,(a+z) =$$
$$\text{Log}\,a + \frac{2}{\text{M}}\left\{\frac{z}{2a+z} + \tfrac{1}{3}\left(\frac{z}{2a+z}\right)^3 + \tfrac{1}{5}\left(\frac{z}{2a+z}\right)^5 + \tfrac{1}{7}\left(\frac{z}{2a+z}\right)^7 \&\text{c.}\right\}$$
$$\text{Log}\,a = \frac{m}{\text{M}}\left\{\sqrt[m]{}(a-1) - \tfrac{1}{2}(\sqrt[m]{a}-1)^2 + \tfrac{1}{3}(\sqrt[m]{a}-1)^3 - \tfrac{1}{4}(\sqrt[m]{a}-1)^4\ \&\text{c.}\right\}$$

Also, $a^x = 1 +$
$$\frac{x}{1}\left(\frac{\log a}{\text{M}}\right) + \frac{x^2}{2}\left(\frac{\log a}{\text{M}}\right)^2 + \frac{x^3}{2.3}\left(\frac{\log a}{\text{M}}\right)^3 + \frac{x^4}{2.3.4}\left(\frac{\log a}{\text{M}}\right)^4 \&\text{c.}$$
$$a = 1 +$$
$$\frac{\log a}{\text{M}} + \tfrac{1}{2}\left(\frac{\log a}{\text{M}}\right)^2 + \frac{1}{2.3}\left(\frac{\log a}{\text{M}}\right)^3 + \frac{1}{2.3.4}\left(\frac{\log a}{\text{M}}\right)^4 \&\text{c.}$$
$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} \&\text{c.}$$

Where e = number whose hyperbolic logarithm is 1, or the radix of the system of the common tabular logarithms, which is known to be 2.7182818284.

84. After these formulæ, which will be found convenient upon many occasions, it will be proper to show the method of obtaining the sine of the sum or difference of any two arcs, in the form of a series; which may be done thus:

Sin $(a \pm z) = \frac{\sin a \cos z \pm \cos a \sin z}{r}$, and cos $(a \pm z) = \frac{\cos a \cos z \mp \sin a \sin z}{r}$ (art. 20); but sin $z = z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4}$ &c.; and cos $z = r - \frac{z^2}{2r} + \frac{z^4}{24r^3}$ &c.; whence, by substitution, we shall have

$$\text{Sin }(a+z)=\sin a+$$
$$\frac{\cos a}{r}z-\frac{\sin a}{2r^2}z^2-\frac{\cos a}{2.3\,r^3}z^3+\frac{\sin a}{2.3.4\,r^4}z+\frac{\cos a}{2.3.4.5\,r^5}z^5-\&c.$$
$$\text{Sin }(a-z)=\sin a-$$
$$\frac{\cos a}{r}z-\frac{\sin a}{2r^2}z^2+\frac{\cos a}{2.3\,r^3}z^3+\frac{\sin a}{2.3.4\,r^4}z^4-\frac{\cos a}{2.3.4.5\,r^5}z^5-\&c.$$
$$\text{Cos }(a+z)=\cos a-$$
$$\frac{\sin a}{r}z-\frac{\cos a}{2r^2}z^2+\frac{\sin a}{2.3\,r^3}z^3+\frac{\cos a}{2.3.4\,r^4}z^4-\frac{\sin a}{2.3.4.5\,r^5}z^5-\&c.$$
$$\text{Cos }(a-z)=\cos a+$$
$$\frac{\sin a}{r}z-\frac{\cos a}{2r^2}z^2-\frac{\sin a}{2.3\,r^3}z^3+\frac{\cos a}{2.3.4\,r^4}z^4+\frac{\sin a}{2.3.4.5\,r^5}z^5-\&c.$$

In each of which series, if $\frac{rz^\circ}{\text{R}^\circ}$ be substituted for z, the arc will be expressed in degrees, instead of by its length; r being = radius, and R° = number of degrees in an arc of equal length with the radius; which is $57^\circ.2957795$ or $206264''.8$.

85. The logarithmic sines, cosines, tangents, &c. of any arc, may also be found from the expressions already given for their natural sines, cosines, &c. as follows:

$$\text{Sin } a=a-\frac{a^3}{2.3\,r^2}+\frac{a^5}{2.3.4.5\,r^4}-\frac{a^7}{2.3.4.5.6.7\,r^6}\ \&c.\ (\text{art. } 50)$$
$$=a\left\{1-\frac{a^2}{2.3\,r^2}+\frac{a^4}{2.3.4.5\,r^4}-\frac{a^6}{2.3.4.5.6.7\,r^6}\ \&c.\right\}$$

Whence

$$\text{Log sin } a=\log a+\log\left(1-\frac{a^2}{2.3\,r^2}+\frac{a^4}{2.3.4.5\,r^4}\ \&c.\right)$$

Or, by putting $p=-\left(\frac{a^2}{2.3r^2}-\frac{a^4}{2.3.4.5\,r^4}\ \&c.\right)$ we shall have, $\log\sin a=\log a+\log(1-p)=\log a-\frac{1}{\text{M}}\left\{p+\frac{1}{2}p^2+\frac{1}{3}p^3\ \&c.\right\}$, as is evident by changing $+p$ to $-p$, in the first form of logarithms, art. 82.

And, by restoring the value of p, it will become

$$\text{Log}\sin a = \log a - \frac{1}{M}\left\{\frac{a^2}{2.3\,r^2} + \frac{a^4}{2^2.3.5\,r^4} + \frac{a^6}{3^4.5.7\,r^6}\,\&c.\right\}$$

86. In like manner, $\cos a = r - \frac{a^2}{2r} + \frac{a^4}{2.3.4r^3} - \frac{a^6}{2.3.4.5.6r^5}$ &c. (art. 51) and consequently $\log\cos a = \log r + \log\left(1 - \frac{a^2}{2r^2} + \frac{a^4}{2.3.4\,r^4} - \frac{a^6}{2.3.4.5.6\,r^6}\,\&c.\right)$ which being treated as in the former article, gives

$$\text{Log}\cos a = \log r - \frac{1}{M}\left\{\frac{a^2}{2r^2} + \frac{a^4}{3.4\,r^4} + \frac{a^6}{3^2.5\,r^6}\,\&c.\right\}$$

87. Again, $\tan a = a + \frac{a^3}{3\,r^2} + \frac{2\,a^5}{3.5\,r^4} + \frac{17\,a^7}{3^2.5.7\,r^6}$&c. (art. 52); whence $\log\tan a = \log a + \log\left(1 + \frac{a^2}{3\,r^2} + \frac{2\,a^4}{3.5\,r^4} + \frac{17\,a^6}{3^2.5.7\,r^6}\,\&c.\right)$; and by finding the logarithm of the series, according to the form $\log(1+p)$, we shall have

$$\text{Log}\tan a = \log a + \frac{1}{M}\left\{\frac{a^2}{3\,r^2} + \frac{7\,a^4}{2.3^2.5\,r^4} + \frac{62\,a^6}{3^4.5.7\,r^6}\,\&c.\right\}$$

From which three formulæ we can also readily obtain, by addition and subtraction, the expressions for the logarithmic secants, cosecants, and cotangents.

88. The logarithmic sines, cosines, &c. of the sum or difference of any two arcs may likewise be readily found, in nearly a similar manner with those of the simple arcs, as follows:

$\text{Log}\sin(a+z) = \log\left\{\sin a\cos z + \cos a\sin z\right\} = \log\left\{\sin a\cos z\left(1 + \frac{\cos a\sin z}{\sin a\cos z}\right)\right\} = \log\left\{\sin a\cos z\,(1 + \tan z\cot a\right\} = \log\sin a + \text{L}\cos z + \log(1 + \tan z\cot a)$.

Whence, $\log\sin(a+z) = \log\sin a + \log\cos z + \frac{1}{M}\left\{\tan z\cot a - \frac{1}{2}\tan^2 z\cot^2 a + \frac{1}{3}\tan^3 z\cot^3 a - \&c.\right\}$

And $\log \sin (a - z) = \log \sin a + \log \cos z - \frac{1}{M}\left\{\tan z \cot a + \frac{1}{2} \tan^2 z \cot^2 a + \frac{1}{3} \tan^3 z \cot^3 a + \&c.\right\}$

89. In like manner, $L \cos (a \pm z) = \log\left\{\cos a \cos z \mp \sin a \sin z\right\} = \log\left\{\cos a \cos z\left(1 \mp \frac{\sin a \sin z}{\cos a \cos z}\right)\right\} = \log \cos a + \log \cos z + \log (1 \mp \tan a \tan z)$.

Whence $\log \cos (a + z) = L \cos a + \log \cos z + \frac{1}{M}\left\{\tan a \tan z - \frac{1}{2} \tan^2 a \tan^2 z + \frac{1}{3} \tan^3 a \tan^3 z \ \&c.\right\}$

And $\log \cos (a - z) = \log \cos a + \log \cos z - \frac{1}{M}\left\{\tan a \tan z + \frac{1}{2} \tan^2 a \tan^2 z + \frac{1}{3} \tan^3 a \tan^3 z \ \&c.\right\}$

90. One of the most commodious forms for the tangent may be found from the formula $\frac{\tan (a+z) - \tan a}{\tan (a+z) + \tan a} = \frac{\sin (a+z-a)}{\sin (a+z+a)} = \frac{\sin z}{\sin (2a+z)}$ (art. 30), which, by reduction, gives $\tan (a + z) = \tan a \left\{\frac{1 + \frac{\sin z}{\sin (2a+z)}}{1 - \frac{\sin z}{\sin (2a+z)}}\right\}$.

Whence $\log \tan (a + z) = \log \tan a + \frac{2}{M}\left\{\frac{\sin z}{\sin (2a+z)} + \frac{1}{3}\left(\frac{\sin z}{\sin (2a+z)}\right)^3 + \frac{1}{5}\left(\frac{\sin z}{\sin (2a+z)}\right)^5 \&c.\right.$

From which expressions the formulæ for the logarithmic secants, cosecants, and cotangents may be readily obtained.

91. The logarithmic sines, cosines, &c. of the sum or difference of any two arcs may also be otherwise expressed, as follows:

$$\text{Log} \sin (a + z) = \log \sin a + \frac{1}{M}\left\{\frac{\cos a}{\sin a} z - \frac{1}{2 \sin^2 a} z^2 - \frac{\cos a}{3 \sin^3 a} z^3 - \frac{1 + 2 \cos^2 a}{12 \sin^4 a} z^4 \ \&c.\right\}$$

$$\text{Log} \sin (a - z) = \log \sin a - \frac{1}{M}\left\{\frac{\cos a}{\sin a} z + \frac{1}{2 \sin^2 a} z^2 + \frac{\cos a}{3 \sin^3 a} z^3 + \frac{1 + 2 \cos^2 a}{12 \sin^4 a} z^4 \ \&c.\right\}$$

$$\text{Log} \cos(a+z) = \log \cos a -$$

$$\frac{1}{M}\left\{\frac{\sin a}{\cos a} z + \frac{1}{2\cos^2 a} z^2 + \frac{\sin a}{3\cos^3 a} z^3 + \frac{1+2\sin^2 a}{12\cos^2 a} z^4 \;\&c.\right\}$$

$$\text{Log} \cos(a-z) = \log \cos a +$$

$$\frac{1}{M}\left\{\frac{\sin a}{\cos a} z - \frac{1}{2\cos^2 a} z^2 + \frac{\sin a}{3\cos^3 a} z^3 - \frac{1+2\sin^2 a}{12\cos^2 a} z^4 \;\&c.\right\}$$

92. It is also evident, from the algebraic expressions for the natural sine, cosine, &c. of any arc, given in art. 18, that their logarithmic sines, cosines, &c. will be as below.

Log sin $a = 20 -$ log cosec $a =$ log cos $a +$ log tan $a - 10 = 10 +$ log tan $a -$ log sec $a = 10 +$ log cos $a -$ log cot a.

Log cos $a = 20 -$ log sec $a = 10 +$ log sin $a -$ log tan $a = 10 +$ log cot $a -$ log cosec $a =$ log sin $a +$ log cot $a - 10$.

Log tan $a = 20 -$ log cot $a = 10 +$ log sin $a -$ log cos $a =$ log sin $a -$ log cot $a +$ log cosec $a =$ log cos $a +$ log sec $a -$ log cot a.

Log cot $a = 20 -$ log tan $a = 10 +$ log cos $a -$ log sin $a =$ log cos $a +$ log sec $a -$ log tan $a =$ log sin $a +$ log cosec $a -$ log tan a.

Log sec $a = 20 -$ log cos $a = 10 +$ log tan $a -$ log sin $a = 30 -$ log sin $a -$ log cot $a = 10 +$ log cosec $a -$ log cot a.

Log cosec $a = 20 -$ log sin $a = 10 +$ log cot $a -$ log cos $a = 30 -$ log cos $a -$ log tan $a = 10 +$ log sec $a -$ log tan a.

Log vers $a =$ log $2 + 2$ log sin $\frac{1}{2} a - 10 = 2$ log sin $\frac{1}{2} a - 9.69899700$.

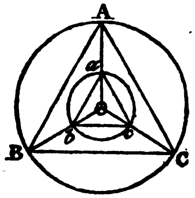
Log supvers $a =$ log $2 + 2$ log cos $\frac{1}{2} a - 10 = 2$ log cos $\frac{1}{2} a - 9.69899700$.

DEMONSTRATIONS OF THE PRINCIPAL THEOREMS IN PLANE TRIGONOMETRY.

Having treated, pretty fully, in the first part of this work, of the practical part of Plane Trigonometry, and its most useful applications in the determination of heights, distances, &c. it will be here proper to give the investigations of the principal theorems from whence those calculations are derived; which are the three following:

THEOREM I.

93. The sides of any plane triangle A B C, are to each other as the sines of their opposite angles; and conversely.



For let the triangle be circumscribed by a circle, and from the centre O, with the radius of the tables, describe the circle *a b c*; and having joined O A, O B, O C, draw the chords *a b*, *b c*, *c a*.

Then, because the angles A O B, B O C, C O A at the centre, are double the angles A C B, B A C, A B C at the circumference, and that the chords *a b*, *b c*, *c a* are twice the sines of half the former of these angles, or of their equals *a* o *b*, *b* o *c*, *c* o *a*, they will be twice the sines of the whole angles A C B, B A C, A B C.

And since o*a*, o*b*, o*c* are equal to each other, being radii of the circle *a b c*, as also O A, O B, O C, the lines *a b*, *b c*, *c a* will be respectively parallel to the opposite sides of the triangle A B, B C, C A.

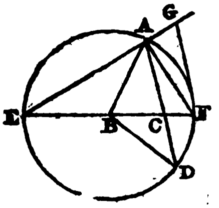
Hence the corresponding triangles O A B, O *a b*, O A C, O *a c* being similar, O A : O *a* :: A B : *a b*, and O A : O *a* :: A C : *a c*; therefore, also, by equality, A B : *a b* :: A C : *a c*; or A B : $\frac{1}{2}$ *a b* :: A C : $\frac{1}{2}$ *a c*.

But $\frac{1}{2}$ *a b* and $\frac{1}{2}$ *a c* have been shown to be the sines of the angles C and B; whence A B : sin ∠ C :: A C : sin ∠ B; or sin ∠ C : A B :: sin ∠ B : A C. Q. E. D.

COR. Since A B : *a b* :: A O : *a* O, or A B : 2 sin ∠ C :: A O : *a* O by similar triangles, it follows that each side of the triangle A B C is to twice the sine of its opposite angle, as the radius of the circumscribing circle is to the radius of the tables.

THEOREM II.

94. The sum of any two sides of a plane triangle A B C, is to their difference, as the tangent of half the sum of their opposite angles is to the tangent of half their difference.



For about one of the angular points B, of the triangle, and with the greater side B A as a radius, describe a circle, meeting B C, produced, in E and F, and A C in D: also join D B, E A, F A, and draw F G parallel to A C.

Then, because B A is equal to B E, it is plain that E C is the sum of the two sides A B, B C, and C F their difference.

And because the outward angle A B E is equal to the sum of the inward angles B A C, B C A, and the angle A F E at the circumference is half the angle A B E at the centre, the angle A F E will be half the sum of the angles B A C, B C A.

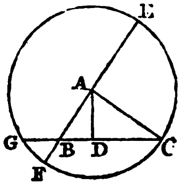
Also, since the angle A C B is equal to the sum of the angles C B D, C D B, or C B D, C A B, the angle F B D will be the difference of the angles B A C, B C A; and the angle F A D at the circumference, or its equal A F G, will be the half difference of those angles.

But the angle E A F, in the semicircle, being a right angle, A G, A E will be the tangents of the angles A F G, A F E to the radius F A.

Hence, since A C, G F are parallel, E C is to C F as E A is to A G; that is, the sum of the sides A B, B C is to their difference, as the tangent of half the sum of their opposite angles B A C, B C A is to the tangent of half their difference. Q. E. D.

THEOREM III.

95. The base of any plane triangle A B C, is to the sum of the sides, as the difference of the sides is to the difference of the segments of the base.



For, about one of the angular points A, of the triangle, as a centre, and with the greater side A C as a radius, describe a circle, meeting A B produced in E and F, and the base C B in G: also draw the perpendicular A D.

Then, because A E, A F are each equal to A C, it is manifest that B E is the sum of the sides A B, A C, and B F their difference.

And because the perpendicular A D, from the centre, bisects C G in D, it is also plain that B G is the difference of the segments of the base C D, D B, or D G, D B.

But since the lines C G, E F in the circle cut each other in B, the rectangle of B E, B F is equal to the rectangle of C B, B G.

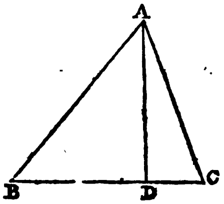
Hence, C B is to B E as B F is to B G; that is, the base B C is to the sum of the sides A B, A C, as the difference of those sides is to the difference of the segments of the base C D, D B. Q. E. D.

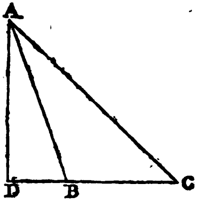
SCHOLIUM. When the perpendicular A D falls without the triangle, the segments of the base must be both reckoned the same way, from the angle C or B, to the foot of that perpendicular.

To these three propositions, which furnish solutions to all the problems that can occur in Plane Trigonometry, it may be proper to add the following, which, when applied to each of the angles, is alone sufficient for that purpose.

THEOREM IV.

96. As twice the rectangle of any two sides of a plane triangle : radius : : sum of the squares of the same two sides — the square of the other side : cosine of the angle included by those sides.





For let AD be perpendicular to the base BC, falling either within or without the triangle, as in the figures.

Then, in the case in which the perpendicular falls within the triangle, we shall have $AC^2 = AB^2 + BC^2 - 2BC \times BD$, or $BD = \frac{AB^2 + BC^2 - AC^2}{2BC}$.

But ABD being a right-angled triangle, it will be as rad : AB :: sin BAD or cos B : BD; or $BD = \frac{AB \cos B}{r}$.

And if this value be substituted in the first equation, it will give $\frac{AB \cos B}{r} = \frac{AB^2 + BC^2 - AC^2}{2BC}$; or $\cos B = r \times \frac{AB^2 + BC^2 - AC^2}{2AB \times BC}$.

Again, if the perpendicular AD falls without the triangle, we shall have, $AC^2 = AB^2 + BC^2 + 2BC \times BD$, or $BD = \frac{AC^2 - AB^2 - BC^2}{2BC}$.

And since ABD is a right-angled triangle, rad : AB :: sin BAD or cos ABD : BD; or $BD = \frac{AB \cos ABD}{r}$.

But ABD being the supplement of ABC, cos ABD $=$ $-$ cos ABC; whence $BD = -\frac{AB \cos ABC}{r}$.

And if this value be substituted in the first equation, we shall have $\frac{AC^2 - AB^2 - BC^2}{2BC} = -\frac{AB \cos ABC}{r}$; or cos ABC

$= r \times \frac{AB^2 + BC^2 - AC^2}{2AB \times BC}$; whence, if these equations be turned into analogies, we shall have $2AB \times BC : \text{rad} :: AB^2 + BC^2 - AC^2 : \cos \angle ABC$.

SCHOLIUM. If the three angles of the triangle be denoted by A, B, C, and their opposite or corresponding sides by *a*, *b*, *c*, the four theorems here demonstrated may be exhibited in general terms, as follows:

$$1.\ \text{Sin}\ A = \frac{a \sin B}{b}$$

$$2.\ \text{Tan}\ \frac{B \sim C}{2} = \frac{b \sim c}{b + c} \cot \tfrac{1}{2} A$$

$$3.\ BD \sim DC = \frac{(b + c) \times (b \sim c)}{a}$$

$$4.\ \text{Cos}\ A = r\left(\frac{b^2 + c^2 - a^2}{2bc}\right).$$

DEMONSTRATIONS OF THE PRINCIPAL THEOREMS IN SPHERICAL TRIGONOMETRY.

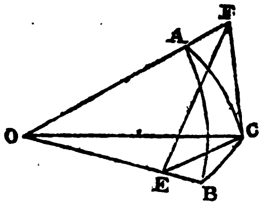
In treating of the practical part of Plane Trigonometry, no distinction was made between right-angled and oblique-angled triangles, on account of the three rules, which are there given, being sufficient to solve every problem that can occur in this branch of the subject, whatever may be the species of the triangle.

But in Spherical Trigonometry, where, from the nature of the subject, the rules of practice become more numerous, it was judged proper to class the various species of these kind of triangles under the three heads of right-angled, quadrantal, and oblique-angled triangles, and to give the rule for each case separately.

The common rules, however, for the various cases in this branch of the science, as well as in the former, may be all derived from the three following theorems, which have the same generality with those mentioned above; and are here demonstrated, from geometrical principles, in a manner equally simple and perspicuous.

THEOREM I.

97. In any right-angled spherical triangle A B C, the sine of either of the legs is to radius, as the tangent of the other leg is to the tangent of its opposite angle.



For let O be the centre of the sphere, and having joined O A, O C, O B, draw C E perpendicular to O B: also, in the plane A O B, draw F E perpendicular to the same line O B, meeting O A, produced, in F; and join F C.

Then, since O E is at right angles to both E C and E F, it will be at right angles to the plane E F C.

And because the plane C O B passes through O E, it will also be perpendicular to the plane E F C; or, which is the same thing, the plane E F C will be perpendicular to the plane C O B.

But the angle A C B being a right angle (by hyp.) the plane C O A or C O F will be perpendicular to the same plane C O B.

Hence the planes E F C, C O F being each perpendicular to the plane C O B, their common section F C will, also, be perpendicular to C O B.

And since E C, E F, which lie in the planes C O B, A O B, are each perpendicular to O B, the angle F E C will be the measure of their inclination, or of the spherical angle C B A.

Also, F C O, C E O, being right angles, F C will be the tangent of the arc A C, and C E the sine of the arc C B, to the radius of the sphere O C.

Hence, F C E being a right-angled plane triangle, right-angled at C, we shall have E C : rad : : C F : tan ∠ F E C; or sin C B : rad : : tan A C : tan ∠ A B C. Q. E. D.

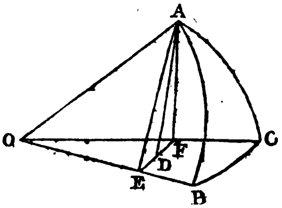
SCHOLIUM. By using the same notation as in the former theorem the present one becomes $r \tan b = \sin a \tan B$; and if $\frac{r^2}{\cot B}$ be put for tan B, we shall have

$$r \sin a = \tan b \cot B,$$

which is the formula for the second case of right-angled spherical triangles.

THEOREM II.

98. In any spherical triangle A B C, whether right-angled or oblique-angled, the sines of the sides are as the sines of their opposite angles; and conversely.



For, let O be the centre of the sphere, and having joined O A, O C, O B, draw A D perpendicular to the plane O B C; also make D E perpendicular to O B, and D F to O C; and join A E, A F.

Then, because A D is perpendicular to the plane O B C, each of the planes A D E, A F D, which pass through A D, will also be perpendicular to that plane.

And since E D is perpendicular to O B, and the plane A D E to the plane O B C, the line A E, which lies in the plane A D E, and is drawn from the same point E, is also perpendicular to O B.

In like manner, because F D is perpendicular to O C, and the plane A F D to the plane O B C, the line F A, which lies in the plane A F D, and is drawn from the same point F, is perpendicular to O C.

Hence the angles A E D, A F D, which measure the inclinations of the planes A O B, A O C, will measure the angles C B A, B C A of the spherical triangle A B C.

Also, A F, being perpendicular to O C, is the sine of the angle A O F, or of the arc A C; and A E, which is perpendicular to O B, is the sine of the angle A O B, or of the arc A B.

But A D E, A F D, being right-angled plane triangles, right-angled at D, we shall have A D = A E sin ∠ A E D, and A D = A F sin ∠ A F D.

Whence, by equality, A E sin ∠ A E D = A F sin ∠ A F D; and consequently, A E : sin ∠ A F D : A F : sin A E D; or sin A B : sin opposite ∠ C : : sin A C : sin opposite ∠ B. Q. E. D.

SCHOLIUM. If the three angles of the triangle A B C be denoted by A, B, C, and their corresponding opposite sides by a, b, c, the proportion obtained above may be represented by the following equation:

$$\text{Sin}\ a \sin B = \sin b \sin A.$$

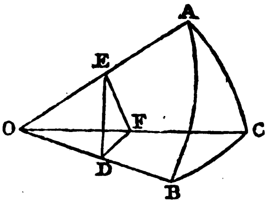
And if A B C be a right-angled triangle, of which C is the right angle, and c the hypothenuse, we shall have

$$r \sin a = \sin c \sin A,$$

which is the formula for the first case of right-angled spherical triangles.

THEOREM III.

99. As the rectangle of the sines of any two sides of a spherical triangle : radius : : rectangle of radius and the cosine of the other side − the rectangle of the cosines of the same two sides : cosine of the angle included by those sides.



For having joined O A, O B, O C, draw F D in the plane O B C, and D E in the plane O A B, each perpendicular to their common section O B, and join E F.

Then, because the angle E D F is the measure of the inclination of the planes O B C, O A B, it is also the measure of the spherical angle A B C or B.

And because $\cos EDF = \frac{r(DE^2+DF^2-EF^2)}{2\,DE\times DF}$, and $\cos EOF = \frac{r(OE^2+OF^2-EF^2)}{2\,OE\times OF}$, or $EF^2 = OE^2 + OF^2 - \frac{2\,OE\times OF\cos EOF}{r}$, if this be substituted in the first equation, we shall have

$$\cos EDF = \frac{r(DE^2+DF^2-OE^2-OF^2)+2\,OE\times OF\cos EOF}{2\,DE\times DF}.$$

But $OE^2 - ED^2$ and $OF^2 - DF^2$ are each equal to OD^2;

whence, cos EDF, or its equal $\cos \angle \mathrm{B} =$

$$\frac{\mathrm{OE} \times \mathrm{OF} \cos \mathrm{EOF} - r \times \mathrm{OD}^2}{\mathrm{DE} \times \mathrm{DF}} = \frac{\mathrm{OE} \times \mathrm{OF} \cdot \cos \mathrm{AC} - r \times \mathrm{OD}^2}{\mathrm{DE} \times \mathrm{DF}}.$$

Or, since $\frac{\mathrm{OE}}{\mathrm{DE}} = \frac{r}{\sin \mathrm{DOE}} = \frac{r}{\sin \mathrm{AB}}$, $\frac{\mathrm{OF}}{\mathrm{DF}} = \frac{r}{\sin \mathrm{DOF}} = \frac{r}{\sin \mathrm{BC}}$, $\frac{\mathrm{OD}}{\mathrm{DE}} = \frac{\cos \mathrm{DOE}}{\sin \mathrm{DOE}} = \frac{\cos \mathrm{AB}}{\sin \mathrm{AB}}$, $\frac{\mathrm{OD}}{\mathrm{DF}} = \frac{\cos \mathrm{DOF}}{\sin \mathrm{DOF}} = \frac{\cos \mathrm{BC}}{\sin \mathrm{BC}}$, if these values be substituted in the former equation, we shall have $\cos \mathrm{B} = \frac{r^2 \cos \mathrm{AC} - r \cdot \cos \mathrm{AB} \cos \mathrm{BC}}{\sin \mathrm{AB} \sin \mathrm{BC}}$, and consequently, $\sin \mathrm{AB} \times \sin \mathrm{BC} : r :: r \cos \mathrm{AC} - \cos \mathrm{AB} \times \cos \mathrm{BC} : \cos \angle \mathrm{B}$. Q. E. D.

Scholium. By using the same letters for the sides and angles of the triangle, as in the two former theorems, the above formula becomes

$$\mathrm{Cos}\ \mathrm{B} = \frac{r^2 \cos b - r \cos a \cos c}{\sin a \sin c}.$$

Which principle being applied successively to all the three angles, furnishes three equations, which are sufficient for resolving all the problems of spherical trigonometry; having the same generality with respect to spherical triangles, that the theorem given in art. 96 has with respect to plane triangles.

A similar formula may also be readily obtained for the cosine of either of the sides in terms of the sines and cosines of the three angles. For since any spherical triangle, whose sides are A, B, C, and opposite angles a, b, c, answers to the polar triangle, whose sides are $180° - \mathrm{A}$, $180° - \mathrm{B}$, and $180° - \mathrm{C}$, and the angles $180° - a$, $180° - b$, $180° - c$, we shall have

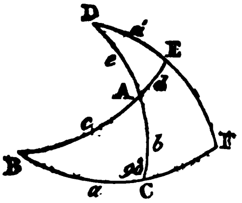
$$\mathrm{Cos}\,(180° - a) = \frac{r^2 \cos(180° - \mathrm{A}) - r \cos(180° - \mathrm{B}) \cos(180° - \mathrm{C})}{\sin(180° - \mathrm{B}) \sin(180° - \mathrm{C})}$$

And because cos $(180^\circ - a) = -\cos a$, cos $(180^\circ - \text{A}) = -\cos \text{A}$, &c. this equation, when reduced, becomes

$$\text{Cos } a = \frac{r^2 \cos \text{A} + r \cos \text{B} \cos \text{C}}{\sin \text{B} \sin \text{C}}$$

Which latter formula resolves immediately the case in which it is required to find a side by means of the three angles, as the former does when it is required to find an angle by means of the three sides.

100. Having obtained by art. 97 the proper formulæ for the solution of the first two cases of right-angled spherical triangles, the rest may be easily derived from them, by means of the complemental triangle, as follows:



Let ABC be a right-angled spherical triangle, and DAE the complemental triangle, formed by producing the sides BA, CA, if necessary, to quadrants.

Then, because the triangle DAE is also right-angled at E, it follows, from theorem 2, that $r \sin a' = \sin e \sin \text{A}$; and since $\sin a' = \cos \text{EF} = \cos \angle \text{B}$, and $\sin e = \cos b$, if these be substituted for a' and e, we shall have, for the 3d case,

$$r \cos \text{B} = \cos b \sin \text{A}.$$

Also, since $r \sin a' = \tan d \cot \text{D}$, by 1st theorem; and $\sin a' = \cos \text{EF} = \cos \text{B}$, $\tan d = \cot c$, and $\cot \text{D}$

$= \cot \mathrm{CF} = \tan a$, we shall have, by substitution, for the 4th case,

$$r \cos \mathrm{B} = \tan a \cot c.$$

In like manner, because $r \sin d = \sin \mathrm{E} \sin \mathrm{D}$, by the 1st theorem, and $\sin d = \cos c$, $\sin e = \cos b$, and $\sin \mathrm{D} = \sin \mathrm{CF} = \cos a$, we shall have, by a similar substitution, for the 5th case,

$$r \cos c = \cos a \cos b.$$

Finally, because $r \sin d = \tan a' \cot \mathrm{A}$, by the 1st theorem, and $\sin d = \cos c$, $\tan a' = \cot \mathrm{EF} = \cot \angle \mathrm{B}$, we shall have, by a like substitution, for the 6th case,

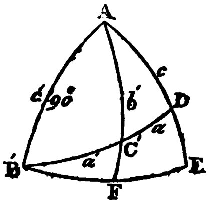
$$r \cos c = \cot \mathrm{A} \cot \mathrm{B}.$$

Hence, all the cases of right-angled spherical triangles being collected together, may be commodiously exhibited at one view, as follows:

$$r \sin a = \sin c \sin \mathrm{A} = \tan b \cot \mathrm{B}$$
$$r \cos \mathrm{B} = \cos b \sin \mathrm{A} = \tan a \cot c$$
$$r \cos c = \cos a \cos b = \cot \mathrm{A} \cot \mathrm{B}.$$

The same forms will also hold in any case, by taking such other sides and angles as are similarly situated with respect to each other.

101. The various cases of quadrantal spherical triangles may also be derived, in a similar way, from the principles above demonstrated, by means of the complemental triangle, as follows:



Let A′ B′ C′ be a quadrantal spherical triangle, and A′ C′ D, or B′ C′ F the complemental triangle, formed by producing the sides A′C′, B′ C′, if necessary, to quadrants.

Then, since by theorem 2 the sines of the sides of any spherical triangle are as the sines of their opposite angles, it follows for the first case, that sin A′ B′ (sin 90°, or rad) : sin ∠ C′ : : sin b' : sin ∠ B′; or

$$r \sin B' = \sin b' \sin C'.$$

Also, because the triangle A′ C′ D is right-angled at D, $r \sin c = \tan a \cot C'A'D$, by 1st theorem; but $\sin c = \sin \angle B'$, $\tan a = \cot a'$, and $\cot C'A'D = \tan B'A'C'$, or $\tan \angle A'$ (B′ A′ D being a right ∠); whence, by substitution, we shall have, for the 2d case,

$$r \sin B' = \cot a' \tan A'.$$

In like manner, because $r \sin a = \sin b' \sin C'A'D$, and $\sin a = \cos a'$, $\sin C'A'D = \cos B'A'C'$, or $\cos \angle A'$ (B′ A′ D being a right ∠), we shall have, by substitution, for the 3d case,

$$r \cos a' = \sin b' \cos A'.$$

Again, because $r \sin a = \tan c \cot C'$, by the 1st theorem, and $\sin a = \cos a'$, $\tan c = \tan A'B'D$, or $\tan B'$, we shall have, by substitution, for the 4th case,

$$r \cos a' = \tan B' \cot C'.$$

Also, because $r \cos C' = \cos c \sin C'A'D$, as has been shown for case 3 of right-angled triangles, and $\cos c = \cos \angle B'$, and $\sin C'A'D = \cos B'A'C'$, or $\cos \angle A'$ (B′ A′ D being a right ∠), we shall have, by substitution, for the 5th case,

$$r \cos C' = \cos A' \cos B'.$$

Lastly, because $r \cos C' = \tan a \cot b'$, as has been shown for the 4th case of right-angled triangles, and $\tan a = \cot a'$, we shall have, by substitution, for the 6th case,

$$r \cos C' = \cot a' \cot b'.$$

Hence, also, by taking, for the sake of uniformity, such sides and angles as correspond with those of the former triangle, all the cases of quadrantal spherical triangles may be exhibited as below:

$$r \sin A' = \sin a' \sin C' = \cot b' \tan B'$$
$$r \cos b' = \sin a' \cos B' = \tan A' \cot C'$$
$$r \cos C' = \cos A' \cos B' = \cot a' \cot b'.$$

Where, by comparing together the similar forms of the two tables, it appears (as well as from the polar triangle, def. 11) that if the sides and angles of any quadrantal spherical triangle be considered, reversedly, as the angles and sides of a right-angled one, the rules for the latter will equally apply to all the cases of the former; observing to change the terms *like* and *unlike* for each other, when the hypothenusal angle is concerned.

102. Next, in order to apply the theorems, above demonstrated, to the solution of the remaining cases of oblique-angled spherical triangles, it will be proper, for the sake of convenience, to present the formulæ already obtained, under all the varieties of which they are susceptible; which, for the case of the sines, in theorem II, are as follows:

$$\text{Sin } A = \frac{\sin a \sin B}{\sin b} = \frac{\sin a \sin C}{\sin c}$$

$$\sin \mathrm{B} = \frac{\sin b \sin \mathrm{A}}{\sin a} = \frac{\sin b \sin \mathrm{C}}{\sin c}$$
$$\sin \mathrm{C} = \frac{\sin c \sin \mathrm{A}}{\sin a} = \frac{\sin c \sin \mathrm{B}}{\sin b}$$
$$\sin a = \frac{\sin c \sin \mathrm{A}}{\sin \mathrm{C}} = \frac{\sin b \sin \mathrm{A}}{\sin \mathrm{B}}$$
$$\sin b = \frac{\sin c \sin \mathrm{B}}{\sin \mathrm{C}} = \frac{\sin a \sin \mathrm{B}}{\sin \mathrm{A}}$$
$$\sin c = \frac{\sin a \sin \mathrm{C}}{\sin \mathrm{A}} = \frac{\sin b \sin \mathrm{C}}{\sin \mathrm{B}}.$$

The three first of which equations furnish the means of determining an angle of the triangle, by means of two of the sides, and the angle opposite to one of them; and the three latter determine a side by means of two of the angles and the side opposite to one of them.

103. In like manner, the general expression obtained by theorem III, when applied to all the sides and angles of the triangle, admits of the following permutations:

$$\cos \mathrm{A} = \frac{r^2 \cos a - r \cos b \cos c}{\sin b \sin c}$$
$$\cos \mathrm{B} = \frac{r^2 \cos b - r \cos a \cos c}{\sin a \sin c}$$
$$\cos \mathrm{C} = \frac{r^2 \cos c - r \cot a \cos b}{\sin a \sin b}$$
$$\cos a = \frac{r \cos b \cos c + \sin b \sin c \cos \mathrm{A}}{r^2}$$
$$\cos b = \frac{r \cos a \cos c + \sin a \sin c \cos \mathrm{B}}{r^2}$$
$$\cos c = \frac{r \cos a \cos b + \sin a \sin b \cos \mathrm{C}}{r^2}.$$

The three first of which equations give the angles by means of the sides; and the three latter give either of the sides, by means of the other two sides and their contained angle.

104. Also, the second general expression, obtained from the same theorem, when applied as before, gives the following formulæ:

$$\text{Cos } a = \frac{r^2 \cos A + r \cos B \cos C}{\sin B \sin C}$$

$$\text{Cos } b = \frac{r^2 \cos B + r \cos A \cos C}{\sin A \sin C}$$

$$\text{Cos } c = \frac{r^2 \cos C + r \cos A \cos B}{\sin A \sin B}$$

$$\text{Cos } A = \frac{\cos a \sin B \sin C - r \cos B \cos C}{r^2}$$

$$\text{Cos } B = \frac{\cos b \sin A \sin C - r \cos A \cos C}{r^2}$$

$$\text{Cos } C = \frac{\cos c \sin A \sin B - r \cos A \cos B}{r^2}$$

The three first of which equations give the sides by means of the angles; and the three latter give either of the angles, by means of the two other angles and their included side.

105. Again, if the value of the cosine of c in the 6th of the 2d set of equations, be substituted for the cosine of c in the 1st of the same set, we shall have $r \cos A \sin c = r \cos a \sin b - \cos C \sin a \cos b$; and by substituting in this equation the value of the sine of c, as given in the 6th of the 1st set of the same formulæ, it will become, after reducing it to its most simple form,

$$\frac{r \cos A}{\sin A} = \cot A = \frac{r \cos a \sin b - \sin a \cos b \cos C}{\sin a \sin C}.$$

106. And if this expression be applied to each of the three angles of the triangle, by using all the permutations of which it is capable, we shall have the six following formulæ:

$$\text{Cot } A = \frac{r \cos a \sin b - \sin a \cos b \cos C}{\sin a \sin C}$$

$$\text{Cot } B = \frac{r \sin a \cos b - \cos a \sin b \cos C}{\sin b \sin C}$$

$$\text{Cot } A = \frac{r \cos a \sin c - \sin a \cos c \cos B}{\sin a \sin B}$$

$$\text{Cot } C = \frac{r \sin a \cos c - \cos a \sin c \cos B}{\sin c \sin B}$$

$$\text{Cot } B = \frac{r \cos b \sin c - \sin b \cos c \cos A}{\sin b \sin A}$$

$$\text{Cot } C = \frac{r \sin b \cos c - \cos b \sin c \cos A}{\sin c \sin A}$$

Which equations determine any two angles of the triangle, when we know the third angle and the two sides by which it is contained.

107. The cotangent of either of the sides may also be determined, in a similar manner, by means of the 1st and 3d set of equations; or more readily by the 1st of the 3d set and the polar triangle. For since $\cot(180^\circ - a) =$

$$\frac{r\cos(180^\circ\text{-}A)\sin(180^\circ - B) - \sin(180^\circ - A)\cos(180^\circ - B)\cos(180^\circ\text{-}C)}{\sin(180^\circ - A)\sin(180^\circ - c)}$$

we shall have $\cot a = \dfrac{r \cos A \sin B + \sin A \cos B \cos c}{\sin A \sin c}$

And this, being applied to each of the three sides, by using all the permutations of which the letters are capable, will give the six following formulæ:

$$\text{Cot } a = \frac{r \cos A \sin B + \cos c \sin A \cos B}{\sin c \sin A}$$

[illegible]

$$\text{Cot } b = \frac{r \sin A \cos B + \cos c \cos a \sin B}{\sin c \sin B}$$

$$\text{Cot } a = \frac{r \cos A \sin C + \cos b \sin A \cos C}{\sin b \sin A}$$

$$\text{Cot } c = \frac{r \sin A \cos C + \cos b \cos A \sin C}{\sin b \sin C}$$

$$\text{Cot } b = \frac{r \cos B \sin C + \cos a \sin B \cos C}{\sin a \sin B}$$

$$\text{Cot } c = \frac{r \sin B \cos C + \cos a \cos B \sin c}{\sin a \sin C}$$

[illegible] cos [illegible] sin α sin Cot B

sin B cot b

Which equations serve to determine any two sides of the triangle, when we know the third side and the two angles between which it is contained (*u*).

108. Of the five classes of formulæ here given, which are sufficient for resolving directly all the cases of spherical trigonometry, the four last deserve particular notice, not only on account of their elegance, but from their possessing the property of showing whether an arc or an angle be greater or less than a quadrant, or 90°.

For the cosine and cotangent of any arc being − in the first quadrant, and + in the second, which is the limit of the sides and angles of every triangle, if care be taken to give to the known quantities, which enter into any of these formulæ, their proper signs, the sign of the result will show the species of the arc or angle sought.

But this cannot be known from the expression of the sine of an arc or angle, as its value and sign are the same both for the arc and its supplement.

109. From these formulæ, which, except those of the first class, are not adapted to logarithmic com-

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(*u*) These formulæ are equally applicable to every species of spherical triangles, whether right-angled, quadrantal, or oblique-angled; and in the two former cases they may be easily resolved into the simple forms before given for the solution of those triangles. The whole doctrine of spherical trigonometry might, therefore, have been deduced from the 2d and 3d theorems above given, or from the 3d alone; but for the sake of the learner, it was judged proper, in the case of right-angled and quadrantal triangles, to follow a mode of investigation which appears something more easy and natural.

putation, we can readily deduce the four elegant theorems commonly known under the name of the *Analogies of Napier*, which are of great use in facilitating the solution of several cases of spherical triangles.

Thus, by a combination of the values of cos A and cos C, given in art. 103, we shall have the two following equations:

$$r \cos \mathrm{A} \sin c = r \cos a \sin b - \cos \mathrm{C} \sin a \cos b$$

$$r \cos \mathrm{B} \sin c = r \cos b \sin a - \cos \mathrm{C} \sin b \cos a$$

And by adding these together, and reducing them, there arises

$$\mathrm{Sin}\, c\, (\cos \mathrm{A} + \cos \mathrm{B}) = (r - \cos \mathrm{C}) \sin (a+b).$$

But since $\frac{\sin b}{\sin \mathrm{B}} = \frac{\sin a}{\sin \mathrm{A}} = \frac{\sin c}{\sin \mathrm{C}}$, we shall have

$$\mathrm{Sin}\, c\, (\sin \mathrm{A} + \sin \mathrm{B}) = \sin \mathrm{C}\, (\sin a + \sin b)$$

$$\mathrm{Sin}\, c\, (\sin \mathrm{A} - \sin \mathrm{B}) = \sin \mathrm{C}\, (\sin a - \sin b)$$

Which two equations, when divided by the preceding one, give

$$\frac{\mathrm{Sin}\, \mathrm{A} + \sin \mathrm{B}}{\cos \mathrm{A} + \cos \mathrm{B}} = \frac{\sin \mathrm{C}}{r - \cos \mathrm{C}} \times \frac{\sin a + \sin b}{\sin (a+b)}$$

$$\frac{\sin \mathrm{A} - \sin \mathrm{B}}{\cos \mathrm{A} + \cos \mathrm{B}} = \frac{\sin \mathrm{C}}{r - \cos \mathrm{C}} \times \frac{\sin a - \sin b}{\sin (a+b)}.$$

And reducing these, by means of the formulæ given in art. 27 and 30, they become

$$\mathrm{Tan}\, \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}\mathrm{C}$$

$$\mathrm{Tan}\, \tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}\mathrm{C}.$$

Which equations determine any two angles of a spherical triangle by means of the two opposite sides and their included angle; and are the same as the practical rule given in the former part of the work.

And if the formulæ, thus obtained, be applied to the polar triangle, by substituting 180°—A, 180°—B,

$180^\circ - a$, $180^\circ - b$, and $180^\circ - c$ in the place of a, b, A, B, C respectively, the result will give the two following analogies.

$$\text{Tan}\ \tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(\text{A}-\text{B})}{\cos\frac{1}{2}(\text{A}+\text{B})}\tan\tfrac{1}{2}c$$

$$\text{Tan}\ \tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(\text{A}-\text{B})}{\sin\frac{1}{2}(\text{A}+\text{B})}\tan\tfrac{1}{2}c.$$

Which equations determine any two sides of a spherical triangle, by means of the opposite angles and their included side; and are the same as the practical rule already given for this purpose.

110. The logarithmic formula for determining either of the angles of a spherical triangle when the three sides are known, may be also obtained from the principles already laid down, as follows:

If the value of the cosine of C (art. 103) be substituted in the formula $2\sin^2\frac{1}{2}\text{C} = r^2 - r\cos\text{C}$ (art. 27), we shall have

$$\frac{2\sin^2\frac{1}{2}\text{C}}{r^2} = 1 - \frac{\cos\text{C}}{r} = \frac{\cos a\cos b + \sin a\sin b - r\cos c}{\sin a\sin b}.$$

Or, since $\cos a\cos b + \sin a\sin b = r\cos(a-b)$, (art. 20) the former expression will become

$$\frac{2\sin^2\frac{1}{2}\text{C}}{r^2} = \frac{r\cos(a-b) - r\cos c}{\sin a\sin b}.$$

But by art. 29, $r\cos(a-b) - r\cos c = 2\sin\frac{1}{2}(c+b-a)\sin\frac{1}{2}(c+a-b)$; whence

$$\frac{\text{Sin}^2\frac{1}{2}\text{C}}{r^2} = \frac{\sin\frac{1}{2}(c+b-a)\sin\frac{1}{2}(c+a-b)}{\sin a\sin b}$$

$$\text{Or, } \sin\tfrac{1}{2}\text{C} = r\sqrt{\frac{\sin\frac{1}{2}(c+b-a)\sin\frac{1}{2}(c+a-b)}{\sin a\sin b}}.$$

In like manner, if the value of the cosine of C be substituted in the formula $2\cos^2\frac{1}{2}\text{C} = r^2 + r\cos\text{C}$ (art. 27), we shall have

$$\frac{2\cos^2\frac{1}{2}\,\mathrm{C}}{r^2} = 1 + \frac{\cos\mathrm{C}}{r} = \frac{r\cos c - \cos a\cos b + \sin a\sin b}{\sin a\sin b}$$

Or since $-(\cos a\cos b - \sin a\sin b) = -r\cos(a+b)$ (art. 20) the former expression will become

$$\frac{2\cos^2\frac{1}{2}\,\mathrm{C}}{r^2} = \frac{r\cos\mathrm{C} - r\cos(a+b)}{\sin a\sin b}.$$

But by art. 29, $r\cos\mathrm{C} - r\cos(a+b) = 2\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+b+c)$; whence

$$\frac{\mathrm{Cos}^2\frac{1}{2}\,\mathrm{C}}{r^2} = \frac{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(a+b-c)}{\sin a\sin b}$$

$$\text{Or, } \cos\tfrac{1}{2}\,\mathrm{C} = r\sqrt{\frac{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(a+b-c)}{\sin a\sin b}}.$$

Or, because $\frac{r\sin\frac{1}{2}\mathrm{C}}{\cos\frac{1}{2}\mathrm{C}} = \tan\frac{1}{2}\,\mathrm{C}$, if the former of these expressions be divided by the latter, we shall have

$$\mathrm{Tan}\,\tfrac{1}{2}\,\mathrm{C} = r\sqrt{\frac{\sin\frac{1}{2}(c+b-a)\sin\frac{1}{2}(c+a-b)}{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(a+b-c)}}.$$

Where either of these three formulæ will determine an angle, when the three sides of the triangle are given.

111. By following the same mode of investigation, the logarithmic expression for determining either of the sides of a spherical triangle, in terms of the three angles, may be obtained as follows:

If the value of the cosine of a (art. 104), be substituted in the formula $\frac{2\sin^2\frac{1}{2}a}{r^2} = 1 - \frac{\cos a}{r}$ (art. 27), we shall have

$$\frac{\sin^2\frac{1}{2}a}{r^2} = \frac{\sin\mathrm{B}\sin\mathrm{C} - \cos\mathrm{B}\cos\mathrm{C} - r\cos\mathrm{A}}{2\sin\mathrm{B}\sin\mathrm{C}}.$$

Or, since $\sin\mathrm{B}\sin\mathrm{C} - \cos\mathrm{B}\cos\mathrm{C} - r\cos\mathrm{A} = -r\cos(\mathrm{B}+\mathrm{C}) - r\cos\mathrm{A}$, the former expression will become

$$\frac{\sin^2 \frac{1}{2} a}{r^2} = \frac{-r \cos (B + C) - r \cos A}{2 \sin B \sin C}.$$

But, by art. 29, $r \cos (B + C) + r \cos A = 2 \cos \frac{1}{2} (A + B + C) \cos \frac{1}{2} (B + C - A)$; whence

$$\frac{\sin^2 \frac{1}{2} a}{r^2} = \frac{-\cos \frac{1}{2} (A + B + C) \cos \frac{1}{2} (B + C - A)}{\sin B \sin C}.$$

(x) Or $\sin \frac{1}{2} a = r \sqrt{\frac{-\cos \frac{1}{2} (A + B + C) \cos \frac{1}{2} (B + C - A)}{\sin B \sin C}}$.

In like manner, if the value of the cosine of a, be substituted in the formula $\frac{2 \cos^2 \frac{1}{2} a}{r^2} = 1 + \frac{\cos a}{r}$, we shall have

$$\frac{\cos^2 \frac{1}{2} a}{r^2} = \frac{\sin B \sin C + \cos B \cos C + r \cos A}{2 \sin B \sin C}.$$

Or, since $\sin B \sin C + \cos B \cos C = r \cos (B - C)$, the former expression will become

$$\frac{\cos^2 \frac{1}{2} a}{r^2} = \frac{r \cos (B - C) + r \cos A}{2 \sin B \sin C}.$$

But, by art. 29, $r \cos (B - C) + r \cos A = 2 \cos \frac{1}{2} (A + B - C) \cos \frac{1}{2} (A + C - B)$, whence

$$\frac{\cos^2 \frac{1}{2} a}{r^2} = \frac{\cos \frac{1}{2} (A + B - C) \cos \frac{1}{2} (A + C - B)}{\sin B \sin C}.$$

And, $\cos \frac{1}{2} a = r \sqrt{\frac{\cos \frac{1}{2} (A + B - C) \cos \frac{1}{2} (A + C - B)}{\sin B \sin C}}$.

Or, because $\frac{r \sin \frac{1}{2} a}{\cos \frac{1}{2} a} = \tan \frac{1}{2} a$, if the former of these expressions be divided by the latter, we shall have

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(x) It may here be observed, that the second member of this equation, though under a negative form, is always affirmative; for $\frac{1}{2} (A + B + C)$ being greater than 90°, its cosine will be negative; and consequently the expression $-\cos \frac{1}{2} (A + B + C)$ will become positive. And since $B + C - A$ can never surpass 180°, or $\frac{1}{2} (B + C - A)$ be greater than 90°, it is plain that $\cos \frac{1}{2} (B + C - A)$ must also be positive.

$$\text{Tan}\ \tfrac{1}{2}\,a = r\sqrt{\frac{-\cos\frac{1}{2}(B+C+A)\cos\frac{1}{2}(B+C-A)}{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A+C-B)}}.$$

Either of which formulæ will determine a side when the three angles of the triangle are given.

112. It may be still further observed, that by taking all the varieties of which these last formulæ for the tangents are susceptible, we shall have

$$\text{Tan}\ \tfrac{1}{2}\,A = r\sqrt{\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+c-b)}{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(a+b+c)}}$$

$$\text{Tan}\ \tfrac{1}{2}\,B = r\sqrt{\frac{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(a+b-c)}{\sin\frac{1}{2}(a+c-b)\sin\frac{1}{2}(a+b+c)}}$$

$$\text{Tan}\ \tfrac{1}{2}\,C = r\sqrt{\frac{\sin\frac{1}{2}(a+c-b)\sin\frac{1}{2}(b+c-a)}{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+b+c)}}$$

$$\text{Tan}\ \tfrac{1}{2}\,a = r\sqrt{\frac{-\cos\frac{1}{2}(B+C-A)\cos\frac{1}{2}(A+B+C)}{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A+C-B)}}$$

$$\text{Tan}\ \tfrac{1}{2}\,b = r\sqrt{\frac{-\cos\frac{1}{2}(A+C-B)\cos\frac{1}{2}(A+B+C)}{\cos\frac{1}{2}(B+C-A)\cos\frac{1}{2}(A+B-C)}}$$

$$\text{Tan}\ \tfrac{1}{2}\,c = r\sqrt{\frac{-\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A+B+C)}{\cos\frac{1}{2}(A+C-B)\cos\frac{1}{2}(B+C-A)}}$$

113. In like manner, by taking all the varieties of which the preceding formulæ for the tangents are susceptible, we shall have

$$\text{Tan}\,\frac{b-a}{2} = \frac{\sin\frac{1}{2}(B-A)}{\sin\frac{1}{2}(B+A)}\tan\tfrac{1}{2}\,c$$

$$\text{Tan}\,\frac{b+a}{2} = \frac{\cos\frac{1}{2}(B-A)}{\cos\frac{1}{2}(B+A)}\tan\tfrac{1}{2}\,c$$

$$\text{Tan}\,\frac{c-b}{2} = \frac{\sin\frac{1}{2}(C-B)}{\sin\frac{1}{2}(C+B)}\tan\tfrac{1}{2}\,a$$

$$\text{Tan}\,\frac{c+b}{2} = \frac{\cos\frac{1}{2}(C-B)}{\cos\frac{1}{2}(C+B)}\tan\tfrac{1}{2}\,a$$

$$\text{Tan}\,\frac{a-c}{2} = \frac{\sin\frac{1}{2}(A-C)}{\sin\frac{1}{2}(A+C)}\tan\tfrac{1}{2}\,b$$

$$\text{Tan}\,\frac{a+c}{2} = \frac{\cos\frac{1}{2}(A-C)}{\cos\frac{1}{2}(A+C)}\tan\tfrac{1}{2}\,b$$

$$\mathrm{Tan}\,\frac{\mathrm{B}-\mathrm{A}}{2}=\frac{\sin\frac{1}{2}(b-a)}{\sin\frac{1}{2}(b+a)}\cot\tfrac{1}{2}\,\mathrm{C}$$

$$\mathrm{Tan}\,\frac{\mathrm{B}+\mathrm{A}}{2}=\frac{\cos\frac{1}{2}(b-a)}{\cos\frac{1}{2}(b+a)}\cot\tfrac{1}{2}\,\mathrm{C}$$

$$\mathrm{Tan}\,\frac{\mathrm{C}-\mathrm{B}}{2}=\frac{\sin\frac{1}{2}(c-b)}{\sin\frac{1}{2}(c+b)}\cot\tfrac{1}{2}\,\mathrm{A}$$

$$\mathrm{Tan}\,\frac{\mathrm{C}+\mathrm{B}}{2}=\frac{\cos\frac{1}{2}(c-b)}{\cos\frac{1}{2}(c+b)}\cot\tfrac{1}{2}\,\mathrm{A}$$

$$\mathrm{Tan}\,\frac{\mathrm{A}-\mathrm{C}}{2}=\frac{\sin\frac{1}{2}(a-c)}{\sin\frac{1}{2}(a+c)}\cot\tfrac{1}{2}\,\mathrm{B}$$

$$\mathrm{Tan}\,\frac{\mathrm{A}+\mathrm{C}}{2}=\frac{\cos\frac{1}{2}(a-c)}{\cos\frac{1}{2}(a+c)}\cot\tfrac{1}{2}\,\mathrm{B}.$$

114. And from these last 12 formulæ may be deduced the following, which serve to find the third side or the third angle, when two sides and the angle opposite to one of them are known.

$$\mathrm{Tan}\,\tfrac{1}{2}\,c=\frac{\sin\frac{1}{2}(\mathrm{B}+\mathrm{A})}{\sin\frac{1}{2}(\mathrm{B}-\mathrm{A})}\tan\tfrac{1}{2}(b-a)$$

$$\mathrm{Tan}\,\tfrac{1}{2}\,c=\frac{\cos\frac{1}{2}(\mathrm{B}+\mathrm{A})}{\cos\frac{1}{2}(\mathrm{B}-\mathrm{A})}\tan\tfrac{1}{2}(b+a)$$

$$\mathrm{Tan}\,\tfrac{1}{2}\,a=\frac{\sin\frac{1}{2}(\mathrm{C}+\mathrm{B})}{\sin\frac{1}{2}(\mathrm{C}-\mathrm{B})}\tan\tfrac{1}{2}(c-b)$$

$$\mathrm{Tan}\,\tfrac{1}{2}\,a=\frac{\cos\frac{1}{2}(\mathrm{C}+\mathrm{B})}{\cos\frac{1}{2}(\mathrm{C}-\mathrm{B})}\tan\tfrac{1}{2}(c+b)$$

$$\mathrm{Tan}\,\tfrac{1}{2}\,b=\frac{\sin\frac{1}{2}(\mathrm{A}+\mathrm{C})}{\sin\frac{1}{2}(\mathrm{A}-\mathrm{C})}\tan\tfrac{1}{2}(a-c)$$

$$\mathrm{Tan}\,\tfrac{1}{2}\,b=\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{C})}{\cos\frac{1}{2}(\mathrm{A}-\mathrm{C})}\tan\tfrac{1}{2}(a+c)$$

$$\mathrm{Cot}\,\tfrac{1}{2}\,\mathrm{C}=\frac{\sin\frac{1}{2}(b+a)}{\sin\frac{1}{2}(b-a)}\tan\tfrac{1}{2}(\mathrm{B}-\mathrm{A})$$

$$\mathrm{Cot}\,\tfrac{1}{2}\,\mathrm{C}=\frac{\cos\frac{1}{2}(b+a)}{\cos\frac{1}{2}(b-a)}\tan\tfrac{1}{2}(\mathrm{B}+\mathrm{A})$$

$$\mathrm{Cot}\,\tfrac{1}{2}\,\mathrm{A}=\frac{\sin\frac{1}{2}(c+b)}{\sin\frac{1}{2}(c-b)}\tan\tfrac{1}{2}(\mathrm{C}-\mathrm{B})$$

$$\mathrm{Cot}\,\tfrac{1}{2}\,\mathrm{A}=\frac{\cos\frac{1}{2}(c+b)}{\cos\frac{1}{2}(c-b)}\tan\tfrac{1}{2}(\mathrm{C}+\mathrm{B})$$

$$\text{Cot } \tfrac{1}{2} \text{ B} = \frac{\sin \frac{1}{2} (a + c)}{\sin \frac{1}{2} (a - c)} \tan \tfrac{1}{2} (\text{A} - \text{C})$$

$$\text{Cot } \tfrac{1}{2} \text{ B} = \frac{\cos \frac{1}{2} (a + c)}{\cos \frac{1}{2} (a - c)} \tan \tfrac{1}{2} (\text{A} + \text{C}).$$

Which formulæ, joined to those of art. 102, for determining a side, or angle, when two sides and an angle opposite to one of them, or two angles and a side opposite to one of them, are given, are sufficient for resolving, logarithmically, every case of spherical triangles.

It may here also be observed, that these equations furnish the means of determining the affections of the sides and angles of spherical triangles, in all the cases which are not necessarily ambiguous, by barely attending to the signs of the quantities of which they are composed.

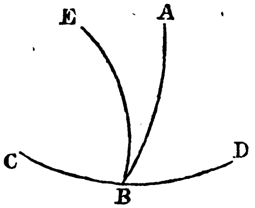
Thus, in the equation $r \cos a = \cos b \cos c$, for right-angled spherical triangles, the 3 sides must be all equal to 90°, or all less, or two of them greater and the third less; as no other combination of them can render the sign of $\cos b \cos c$ like that of $\cos a$, as the equation requires.

Also, in the last analogy for oblique-angled spherical triangles, art. 113, as cot $\frac{1}{2}$ B and cos $\frac{1}{2}$ $(a \sim c)$ are both positive, tan $\frac{1}{2}$ (A + C) and cos $\frac{1}{2}$ $(a + c)$ must have the same sign; hence, half the sum of any two sides is of the same kind as half the sum of their opposite angles: which consideration will sometimes take away an ambiguity that might otherwise arise, in cases where the quantity sought is to be determined by means of a sine.

SPHERICAL THEOREMS.

THEOREM I.

115. If two arcs of circles meet each other, they make two angles, which are, together, equal to two right angles, or 180°.



Let the arc AB meet the arc CD in the point B; then will the two $\angle^s$ ABC, ABD be equal to two right angles.

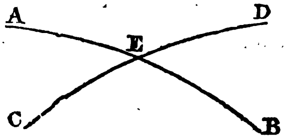
For, suppose the arc EB to be perpendicular to CD, then the $\angle^s$ EBC, EBD are right angles.

And since the $\angle$ EBD is equal to the $\angle^s$ EBA, ABD, the three $\angle^s$ EBC, EBA, ABD are equal to two right $\angle^s$.

But the two $\angle^s$ EBC, EBA are equal to the $\angle$ ABC, whence the two $\angle^s$ ABC, ABD are also equal to two right angles. Q. E. D.

THEOREM II.

116. If two arcs of circles intersect each other, the vertical or opposite angles will be equal.



Let the two arcs AB, CD intersect each other in E, then will the $\angle$ AEC be equal to DEB, and AED to CEB.

For since the arc AE meets the arc CD, the $\angle^{s}$ AEC, AED are, together, equal to two right $\angle^{s}$ (theo. I).

And because the arc DE meets the arc AB, the $\angle^{s}$ DEB, DEA are also equal to two right $\angle^{s}$.

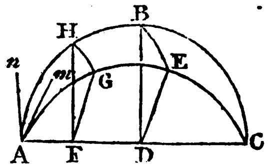
Whence the sum of the $\angle^{s}$ AEC, AED is equal to the sum of the $\angle^{s}$ DEB, DEA.

And if the $\angle$ AED, which is common, be taken away, the remaining $\angle$ AEC will be equal to the remaining $\angle$ DEB. And in the same manner it may be shown, that the $\angle$ AED is equal to CEB. Q. E. D.

COR. If two arcs of circles intersect each other, the $\angle^{s}$ about the point of intersection are, together, equal to four right $\angle^{s}$.

THEOREM III. (*y*)

117. An angle made by any two great circles of the sphere is equal to the angle of inclination of the planes of those circles.



Let BAE be a spherical triangle, made by the two great circles CBA, CEA; then will this angle be equal to the angle of inclination of the planes of those circles.

For take the arcs AB, AE each equal to 90°, and through the points B, E draw the arc of a great circle BE, and from D, the centre of the sphere, draw DB, DE.

(*y*) Any two great circles of the sphere, ABC, AEC, which pass through the poles A, C of another great circle BE, cut all the parallels HG, BE into similar arcs, or such as contain the same number of degrees.

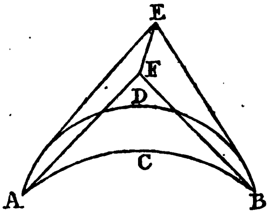
Then because AB, AE are quadrants, the lines DB, DE are each perpendicular to the common section AC; and consequently BDE is the ∠ of inclination of the planes CBA, CEA.

But since DB, DE are equal, being radii of the sphere, the ∠BDE, which is measured by the arc BE, is equal to the ∠BAE, which is measured by the same arc.

And if FH be drawn in the plane CBA, and FG in the plane CEA, each perpendicular to the common section AC, the ∠HFG, which is equal to ∠BDE, will also be equal to the ∠BAE. Q. E. D.

COR. The ∠BAE made by two great circles of the sphere BA, EA, is equal to the $\angle nAm$, formed by two tangents drawn from the angular point A, one in each plane, these tangents being each perpendicular to the diameter AC.

SCHOLIUM. As the $\angle^{s}$ BAE, BCE, formed by the intersections of two great circles of the sphere, are equal, so it may be easily proved that if the arcs ACB, ADB of any two circles, whether great or small, intersect each other, either in a plane, or on the surface of a sphere, the opposite $\angle^{s}$ of the lunule BAD, ABD will be equal.



For draw AE, AF touching the arcs AD, AC in A, and BE, BF touching the arcs BD, BC in B, and meeting each other in E and F: also join EF.

Then since EA, EB are tangents to the circle ADB, and meet in the point E, they are equal.

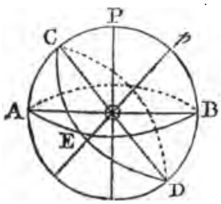
And, because FA, FB are also tangents to the circle ACB, and meet in the point F, they are equal.

Hence the sides AE, AF of the △ AFE being equal to the sides BE, BF of the △ BFE, and the base EF common, the ∠ EAF will be = the ∠ EBF.

But these $\angle^s$ are equal to the curvilinear $\angle^s$ BAD, ABD (*cor.*); whence the latter are also equal.

THEOREM IV.

118. The distance of the poles of any two great circles of the sphere is equal to the angle of inclination of the planes of those circles.



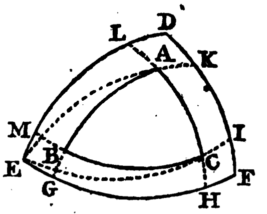
Let AEB, CED be two great circles, and P, *p* their poles; then will the arc P*p* be equal to the angle of their inclination AOC, or BOD.

For since P is the pole of the circle AEB, and *p* of CED, the arc PA will be equal to *p*C, being each quadrants, or 90°.

And if PC, which is common to each, be taken away, the remaining arc P*p*, which is the distance of the two poles, is equal to CA, which is the measure of the ∠ of inclination AOC. Q. E. D.

THEOREM V.

119. If arcs be described from the three angular points of any spherical triangle A B C, as poles, the sides and angles of the triangle D E F, formed by their intersection, will be the supplements of the angles and sides of the former, and vice versâ.



For let the sides of the △ A B C be produced, if necessary, till they meet the sides of the △ D E F, in the points G, H, I, K, L, M; and draw the arcs E A, E C.

Then, since the points A, C are the poles of the arcs E F, E D, the arcs C E, A E will be quadrants, or each 90°; and consequently E is the pole of A C.

In like manner, it may also be shown that F is the pole of A B, and D the pole of B C.

Since, therefore, E is the pole of A H, and F the pole of A G, the arcs E H, F G will be each quadrants, and their sum $EH+FG$ or $EF+GH=180°$.

But A being the pole of E F, the arcs A G, A H are also each quadrants; and consequently G H is the measure of the ∠ B A C or A.

Whence, $EF+GH$ being $=180°$, $EF+\angle A$ is also $=180°$; or $EF=180°-\angle A$.

And in the same manner it may be shown, that $FD=180°-\angle B$, and $DE=180°-\angle C$.

Again, C being the pole of MD, and B the pole of ID, the arcs CM, BI are each quadrants; and their sum $CM + BI$ or $MI + BC = 180°$.

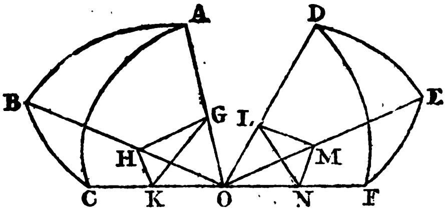
But the arc MI having the point D for its pole, is the measure of the $\angle EDF$ or D; whence $\angle D + BC$ is also $= 180°$; or $\angle D = 180° - BC$.

And in the same way it may be shown, that $\angle E = 180° - AC$, and $\angle F = 180° - AB$.

Hence, also, reciprocally, $\angle A = 180° - EF$, $\angle B = 180° - FD$, $\angle C = 180° - DE$; and $BC = 180° - \angle D$, $AC = 180° - \angle F$, $AB = 180° - \angle F$. Q. E. D.

THEOREM VI.

120. If the three sides of one spherical triangle be equal to the three sides of another, each to each, the angles which are opposite to the equal sides will be equal.



Let ABC, DEF be two spherical triangles, having the side $AB = DE$, $AC = DF$, and $BC = EF$; then will $\angle A = \angle D$, $\angle B = E$, and $\angle C = \angle F$.

For take any two equal distances OG, OL, on the equal radii OA, OD; and in the planes OAC, OAB, draw GK, GH each perpendicular to OA; and in the planes DOF, DOE draw LN, LM each perpendicular to OD.

Then, because the side OG of the Δ OHG $=$ OL of the Δ OLM, the $\angle GOK = \angle LON$ (being measured by the equal arcs AC, DF), and ∠OGK = ∠OLN (being right ∠s), the side GK will be = to LN, and OK to ON.

In like manner, because the side OG of the △OHG = OL of the △OML, the ∠GOH = ∠LOM (being measured by the equal arcs AB, DE), and the ∠OGH = ∠OLM (being right ∠s), the side GH will be = to LM, and OH to OM.

Since, therefore, the sides OK, OH of the △OHK, are = the sides ON, OM of the △OMN, and the ∠HOK = ∠MON (being measured by the equal arcs BC, EF), the side KH will also be = to NM.

Hence, the three sides of the △GHK being equal to the three sides of the △LMN, the ∠s KGH, NLM, which are opposite to the equal sides KH, NM will also be equal.

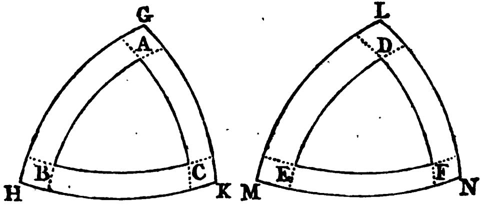
But the ∠KGH, which is the inclination of the two planes AOC, OAB is = the spherical ∠BAC, and the ∠NLM, which is the inclination of the two planes DOF, DOE, is = the spherical ∠EDF; whence the spherical ∠s BAC, EDF are also equal.

And, if two equal distances be taken in the radii OB, OE, or OC, OF, and perpendiculars be drawn from their extremities in the other planes, as before, it may be shown, in a similar manner, that the ∠ABC is = ∠DEF, and ∠ACB = ∠DFE (z). Q. E. D.

(z) Most of the trigonometrical writers have attempted to prove this, and several other propositions in Spherics, by means of laying one triangle upon the other, as in plane geometry. But this method, in several cases, is not exact, as there may be two spherical triangles, as well as two solids, or two olid angles, which are

THEOREM VII.

121. If the three angles of one spherical triangle be equal to the three angles of another, each to each, the sides which are opposite to the equal angles will be equal.



Let ABC, DEF be two spherical △s, having ∠ A = ∠ D, ∠ B = ∠ E and ∠ C = ∠ F; then AB = DE, BC = EF, and CA = FD.

For, about the angular points of the two △s describe the supplemental or polar △s GHK, LMN.

Then, because the ∠ A, B, C are, respectively, = to the ∠s D, E, F, the sides HK, KG, GH which are the supplements of the former, will be = sides MN, NL, LM which are the supplements of the latter.

And since the three sides of the △ GHK are equal to the three sides of the △ LMN, each to each, the ∠s G, H, K will be respectively equal to the ∠s L, M, N.

But the ∠s G, H, K being the supplements of the sides BC, CA, AB, and the ∠s L, M, N the supplements of EF, FD, DE, the side AB will be = DE, BC = EF, and CA = FD. Q. E. D.

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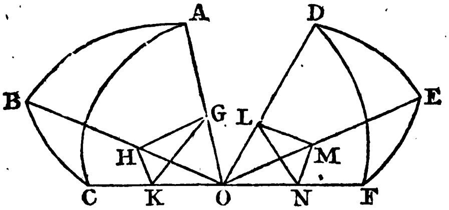
equal in all their constituent parts, and yet are not superposable, or equal by coincidence; as is shown by *Legendre*, prop. 11. b. vii. of his Geometry; where he also observes (note 1.) that Dr. Simpson, in objecting to the demonstration of prop. 28, b. xi. of Euclid, has himself fallen into this mistake, by founding his demonstration upon a coincidence that does not exist.

THEOREM VIII.

122. If two sides and the included angle of one spherical triangle be equal to two sides and the included angle of another, each to each, the remaining sides and angles will be equal.



Let ABC, DEF be two spherical △^{s} having the side AB = DE, AC = DF, and ∠BAC = ∠EDF; then will the side BC = EF, ∠ABC = ∠DEF, and ∠ACB = ∠DFE.

For take any two equal distances OG, OL on the equal radii OA, OD; and in the planes OAC, OAB draw GK, GH each perpendicular to OA; and in the planes DOF, DOE, draw LN, LM each perpendicular to OD.

Then, because the side OG of the △ OKG is = to OL of the △ ONL, the ∠ GOK = ∠ LON (being measured by the equal arcs AC, DF), and ∠ OGK = ∠ OLN (being right ∠^{s}), the side GK will be = to LN, and OK to ON.

In like manner, because the side OG, of the △ OHG, is = OL, of the △ OML, the ∠ GOH = ∠ LOM (being measured by the equal arcs AB, DE), and the ∠ OGH = ∠ OLM (being right ∠^{s}), the side GH will be = to LM, and OH to OM.

Since, therefore, the sides GK, GH, of the △ GHK, are equal to LN, LM of the △ LMN, and ∠ KGH =

∠ N L M (being the inclinations of the planes which form the equal ∠s B A C, E D F), the side K H will also be equal to N M.

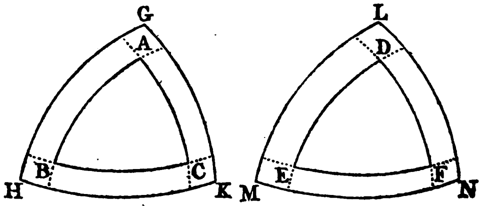
Hence the three sides of the △ H K O being equal to the three sides of the △ O M N, the ∠s H O K, M O N, which are opposite to the equal sides K H, N M, are also equal.

But these ∠s being at the centre of the sphere, are measured by the arcs B C, E F at the circumference, which subtend them; whence the side B C is equal to E F.

And since the three sides of the △ A B C are equal to the three sides of the △ D E F, it follows, from the last proposition, that the ∠ A B C = ∠ D E F, and ∠ A C B = ∠ D F E. Q. E. D.

THEOREM IX.

123. If a side and the two adjacent angles of one spherical triangle be equal to a side and the two adjacent angles of another, each to each, their remaining sides and angles will be equal.



Let A B C, D E F be two spherical △s, having the side B C = E F, ∠ B = ∠ E, and ∠ C = ∠ F: then will side A B = D E, A C = D F, and ∠ A = ∠ D.

For, about the angular points of the two △s, describe the supplemental or polar △s G H K, L M N.

Then, because the side B C = E F, the $\angle^s$ G, L, which are their supplements, are also equal.

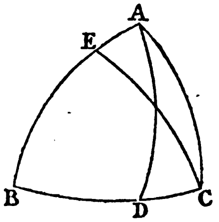
And because the $\angle^s$ B, C are = $\angle^s$ E, F, the sides G K, G H, which are the supplements of the former, are = sides L M, L N, which are the supplements of the latter.

But since the sides G H, G K and the included ∠ G, of the △ G H K, are = sides L M, M N and the included ∠ L, of the △ L M N, the side H K will be = M N, ∠ H = ∠ M, and ∠ K = ∠ N.

Hence, also, ∠ A = ∠ D, being the supplements of H K, M N, and A B, A C, being supplements of the $\angle^s$ K, H, are = D E, D F, which are the supplements of $\angle^s$ N, M. Q. E. D.

THEOREM X.

125. The angles at the base of an isosceles spherical triangle are equal; and if the angles at the base are equal, their opposite sides are equal.



Let A B C be an isosceles △, having the side A B = to A C; then will the ∠ A B C be equal to ∠ A C B.

For, bisect the base B C in D, and through the points A, D draw the arc A D.

Then, because the two sides A B, B D of the △ A D B, are equal to A C, C D of the △ A D C, and the side A D is common to each, the ∠ A B C will be equal to ∠ A C B.

Again, if ∠ ABC be equal to ∠ ACB, the side AB will be equal to AC.

For if not, let AB be the greater, and take BE equal to AC, and draw the arc CE.

Then, because the two sides EB, BC, and the included ∠ EBC, of the △ BEC, are equal to AC, CB, and the included ∠ ACB, of the △ BAC, the ∠ ECB will be equal to ∠ ABC.

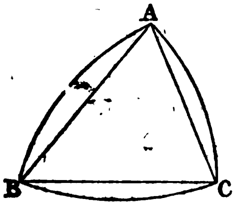
But ∠ ABC is equal to ∠ ACB (by hyp.); whence ∠ ECB is also equal to ∠ ACB, the less to the greater; which is impossible.

The side AB is, therefore, not greater than AC; and in the same manner it may be shown that it is not less; whence they are equal. Q. E. D.

Cor. A perpendicular drawn from the vertex of an isosceles spherical △ to the base, bisects both the base and the vertical angle, except when the two equal sides are quadrants; in which case there are an indefinite number of perpendiculars.

THEOREM XI.

125. The sum of any two sides of a spherical triangle is greater than the third side; and the difference of any two sides is less than the third side.



Let ABC be a spherical triangle; then will the sum of any two sides A B, A C be greater than B C; and the difference of A B, A C less than B C.

For draw the chords A B, A C, B C, which will fall within the sphere.

Then, since these chords form a plane △, the sum of any two of its sides will be greater than the third side.

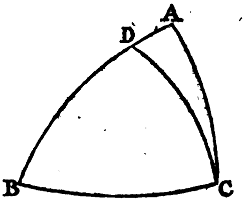
And because, in the same circle, the greater chord subtends the greater arc, the sum of any two sides A B + A C, of the spherical △ A B C, is greater than the third side B C.

Also, since the side A B is less than B C + A C, if A C be taken from each of them, the difference of A B and A C is less than B C. Q. E. D.

Cor. The shortest distance between any two points on the surface of a sphere is the arc which passes through those points.

THEOREM XII.

126. The greater side of any spherical triangle is opposite to the greater angle, and the least side to the least angle; and conversely.



Let A B C be a spherical △; then if ∠ C be greater than ∠ B, A B will be greater than A C; and if A B be greater than A C, ∠ C will be greater than ∠ B.

For through the point C draw the arc C D, making ∠ B C D equal to ∠ D B C.

Then, in the △ A D C, the sum of the sides A D + D C is greater than A C (theo. 11).

But ∠ B C D being equal to ∠ D B C (by const.) the side D C is equal to D B (theo. 10); whence, also, A D + D B or A B is greater than A C.

Again, if A B be greater than A C, ∠ A C B will be greater than ∠ A B C.

For, if not, it must be either equal or less.

But angle A C B cannot be equal to ∠ A B C; for, in this case, A B (theo. 10) would be equal to A C, which it is not.

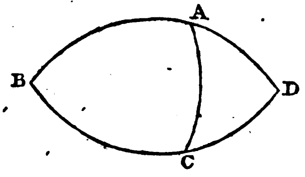
Neither can ∠ A C B be less than ∠ A B C; for, in that case, A B (by 1st part prop.) would be less than A C, which it is not.

Whence ∠ A C B being neither equal to nor less than ∠ A B C, must be greater than it.

And in a similar manner it may be shown that the least side is opposite to the least ∠, and the least ∠ to the least side. Q. E. D.

THEOREM XIII.

127. The sum of the three sides of any spherical triangle is less than the circumference of a circle, or 360°; and the difference of any two sides is less than 180°.



Let ABC be a spherical △; then will the sum of its three sides AB + BC + CA be less than 360°; and the difference of AB, AC less than 180°.

For, produce the sides BA, BC till they meet in the opposite point of the sphere at D.

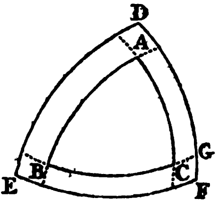
Then, since the arcs BAD, BCD are semicircles; the sum of the arcs BA + BC + DA + DC is equal to a circle, or 360°.

But the sum of the two sides DA + DC is greater than AC (theo. 11); whence the sum of the three sides BA + BC + AC is less than a circle, or 360°.

Also, the difference of AB, AC being less than BC (theo. 11), and BC less than 180°, AB ∼ AC must be less than 180°. Q. E. D.

THEOREM XIV.

128. The sum of the three angles of every spherical triangle is greater than two right angles, or 180°, and less than six, or 540°.



Let ABC be a spherical △; then will the sum of its $\angle^s$ A + B + C be greater than 180°, and less than 540°.

For, about the angular points A, B, C describe the supplemental or polar △ DEF.

Then, since ∠ A = 180° − EF, ∠ B = 180° − FD, and ∠ C = 180° − DE, their sum ∠ A + ∠ B + ∠ C will be = 540° − (EF + FD + DE).

But the sum of the sides EF + FD + DE being less than 360° (theo. 13), if this be taken from 540°, the remainder will be greater than 180°.

Whence, also, ∠ A + ∠ B + ∠ C, which is equal to this difference, will be greater than 180°.

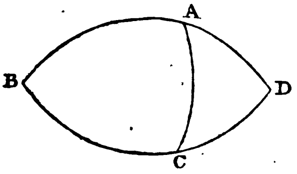
And because each ∠ of the △ is less than 180°, their sum A + B + C must be less than 540°. Q. E. D.

COR. The sum of any two $\angle^s$ of a spherical △, is greater than the supplement of the third angle.

For ∠ A + ∠ B + ∠ C being greater than two right $\angle^s$, or than ∠ ACB + ∠ ACG, if ∠ ACB or C be taken away, the sum of the remaining $\angle^s$ A + B will be greater than ∠ ACG (*a*).

THEOREM XV.

129. If the sum of any two sides of a spherical triangle be equal to, greater, or less than a semicircle, the sum of their opposite angles will, accordingly, be equal to, greater, or less than two right angles; and conversely.



(*a*) Mr. Vince, in his Treatise of Trigonometry (p. 112, prop. 17), has endeavoured to prove that the sum of any two $\angle^s$ of a spherical △ is greater than the third. But this is not true, except in right-angled $\triangle^s$, as may be easily shown, either by partial examples, or by a general investigation. As an instance of the former kind, let ABC (fig. to the prop.) be an isosceles △, having its equal $\angle^s$ A and B each less than 45°, then will their sum be less than 90°; and consequently the remaining ∠ C must be greater than 90°; or

Let ABC be a spherical △; then if AB + AC be equal to, greater, or less than a semicircle, or 180°, the sum of the ∠s B + C will be equal to, greater, or less than 180°.

For produce the sides BA, BC till they meet in the opposite point of the sphere at D.

Then, if AB + AC be equal to the semicircle BAD, the side AC will be = AD, and the ∠ ACD = ∠ D or B (theo. 10).

But ∠ ACD + ∠ ACB = two right ∠s or 180°; whence, also, ∠ ACB + ∠ B = 180°.

Again, if AB + AC be greater than the semicircle BAD, the side AC will be greater than AD; and ∠ D or B greater than ∠ ACD (theo. 12).

But ∠ ACD + ∠ ACB = two right ∠s, as before; whence, also, ∠ ACB + ∠ B is greater than 180°.

Lastly, if AB + AC be less than the semicircle BAD, the side AC will be less than AD, and ∠ D or B less than ∠ ACD (theo. 12).

But ∠ ACD + ∠ ACB being = two right ∠s, or 180°, the sum of the ∠s ACB + B is less than 180°.

And, in a similar manner, it may be shown, that if the sum of the two ∠s B + C be equal to, greater, or less than 180°, the sum of their opposite sides AB + AC, will also be equal to, greater, or less than 180°.

Q. E. D.

COR. 1. If each side of a spherical △ be equal to,

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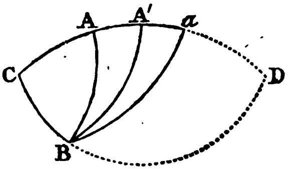
otherwise the sum of the three ∠s of the △ would be less than 180°; which being contrary to the above proposition, is a sufficient proof that the principle is erroneous.

greater, or less than 180°, each of the $\angle^s$ will, accordingly, be right, obtuse, or acute; and conversely.

Cor. 2. Half the sum of any two sides of a spherical △ is of the same kind as half the sum of their opposite angles.

THEOREM XVI.

130. In any right-angled or quadrantal spherical triangle, the legs, or sides, are of the same kind as their opposite angles; and conversely.



Let ABC, A′BC, or *a*BC be a right-angled spherical △, of which C is the right ∠; then will the leg AC, A′C, or *a*C be like its opposite ∠.

For let A′C be equal to a quadrant, AC less than a quadrant, and *a*C greater; and through the points A, A′, *a*, and B draw the circles AB, A′B, and *a*B.

Then, because A′ is the pole of the circle CBD, the ∠A′BC is a right ∠, or 90° (def.); and, consequently, the ∠ABC is less than 90°, and the ∠*a*BC greater, agreeing with the opposite leg A′C, AC, or *a*C.

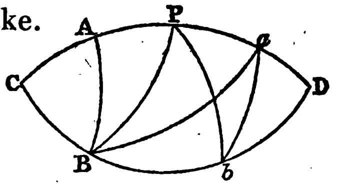
On the contrary, if A′BC be a right ∠, A′ will be the pole of CBD, and A′C will be a quadrant; whence, also, if the ∠ABC be less than a right ∠, and the ∠*a*BC greater, the opposite leg AC will be less than the quadrant A′C, and *a*C greater. Q. E. D.

The same will also hold if the △ be quadrantal; for its sides and $\angle^s$ being the supplements of the $\angle^s$

and legs of the polar △, which, in this case, is right $\angle^{d}$, the similarity will be the same as before.

THEOREM XVII.

131. In any right-angled spherical △ the hypothenuse is less or greater than 90°, according as the two legs, or the two angles, or a leg and its adjacent angle, are like or unlike.



1st. If the △ A B C, right-angled at C, have its legs C A, C B each less than 90°, the hypothenuse A B will be less than 90°.

For make C P equal to a quadrant, and through the points P, B draw the arc of a great circle P B.

Then, because P is the pole of the great circle C B D, the arc P B is a quadrant, or 90° (def.)

And since, in the right $\angle^{d}$ $\triangle^{s}$ P C B, A D B, the leg C B is less than 90°, and D B greater, the ∠ C P B or A P B is also less than 90°, and the ∠ D A B, or P A B greater than 90° (theo. 16).

But the less side of every △ being opposite to the less ∠, the hypothenuse A B is less than 90°, or than the quadrant P B.

2dly. If the △ *a* C *b* have its legs C *a*, C *b* each greater than 90°, the hypothenuse *a b* will, in this case also, be less than 90°.

For produce C *a*, C *b* till they meet at D, which will be a right angle, and through the points P, *b* draw the quadrant P *b*.

Then, since the legs D *a*, D *b* are each less than 90°, it may be shown, as before, that the hypothenuse *a b*, which is common to both the △^{s} *a* C *b*, *a* D *b*, is less than 90°, or than the quadrant P *b*.

3dly. If the △ *a* C B have one leg C B less than 90°, and the other C *a* greater, the hypothenuse *a* B will be greater than 90°.

For, since in the right ∠^{d} △^{s} *a* C B, P D B, the leg C B is less than 90°, and D B greater, the ∠ C *a* B, or P *a* B, is less than 90°, and the ∠ D P B, or *a* P B greater (theo. 12); whence, also, *a* B is greater than 90°, or than the quadrant P B.

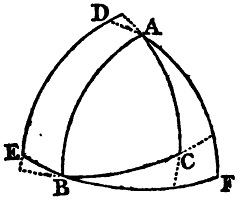
Again, the ∠^{s} in either of the △^{s} A B C, *a b* C, or *a* B C, being of the same kind as their opposite legs (theo. 16), it follows, that the hypothenuse A B, *a b*, or *a* B is less or greater than 90°, according as the two oblique ∠^{s}, of the △ to which it belongs, are like or unlike.

And because a leg and an ∠ in each of these △^{s} are of the same kind as the two legs (the other leg being like its opp. ∠), it is plain that the hypothenuse A B, *a b*, or *a* B is also less or greater than 90°, according as either leg and its adjacent ∠ are like or unlike. Q. E. D.

COR. It follows, reversedly, from this proposition, that in any right-angled spherical △, either leg is less or greater than 90°, according as its adjacent ∠ and the hypothenuse, or the other leg and the hypothenuse, are like or unlike.

Also, that either of the oblique angles is acute or obtuse, according as its adjacent leg and the hypothenuse, or the other ∠ and the hypothenuse, are like or unlike.

Scholium. This proposition and its corollaries will also hold for any quadrantal spherical △, observing to substitute the hypothenusal ∠ for the hypothenuse, and the terms greater or less for less or greater.



For the sides and angles of the quadrantal △ A B C are, evidently, like or unlike, according as the angles or legs of the right ∠^{d} polar △ D E F, which are their supplements, are like or unlike.

But the hypothenusal ∠ C, being the supplement of the hypothenuse D E, will consequently be greater than 90° when D E is less, and less than 90° when D E is greater; which is, therefore, the only change that takes place in the proposition.

OF THE
STEREOGRAPHIC PROJECTION OF THE SPHERE.

The stereographic projection of the sphere, is such a representation of the various parts of its surface, on the plane of one of its great circles, as would be formed by lines drawn from the pole of that circle to every point of the figure to be delineated.

Or, if taken in an optical sense, it is a view of the points and circles of the sphere, as they would appear on a transparent plane, passing through the centre, to an eye placed at one of the extremities of a diameter, drawn perpendicular to that plane.

The place of the eye, is called *the projecting point*, and the plane, on which the points and circles of the sphere are to be represented, is called *the plane of projection.*

The primitive circle, is that which lies in the plane of projection; being the one to which all the other circles and points of the sphere are referred.

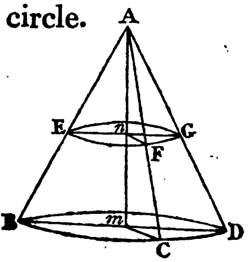
A right circle, is that which, passing through the eye, has its plane perpendicular to the plane of the primitive; and, being seen edgewise, is projected into a right line.

A parallel circle, is that which is parallel to the primitive; and *an oblique circle* is that which is seen obliquely by the eye.

It is also to be observed, that the projection of any point of the sphere, is that point in the plane of projection, which is cut by a right line drawn from the original point to the eye. And that lines flowing to the projecting point, or place of the eye, from every point of the circumference of a circle, form the convex surface of a cone.

LEMMA.

132. If a cone be cut by a plane parallel to its base, the section will be a circle.



Let ABCD be a cone, either right or oblique, and EFG a section parallel to its base BCD; then will EFG be a circle.

For let the planes AC*m*, AD*m* pass through the axis A*m* of the cone, meeting the section in the points F, G, *n*;

Then, because the section EFG is parallel to the base BCD, and the planes C*n*, D*n* meet them, *n*F will be parallel to *m*C, and *n*G to *m*D.

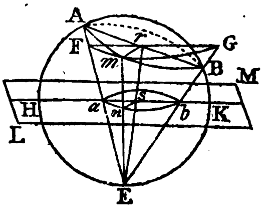
And, because the $\triangle^s$, formed by these lines, are similar, A*m* : A*n* : : *m*C : *n*F or as *m*D : *n*G.

But *m*C is equal to *m*D, being radii of the same circle; whence, also, *n*F is equal to *n*G.

And the same may be shown for any other lines drawn from the point *n* to the circumference of the section EFG, which is therefore a circle. Q. E. D.

THEOREM I.

133. Every circle of the sphere, which does not pass through the poles of the primitive, is projected into a circle.



Let A*m*B be a circle to be projected on the plane LM, which passes through the centre of the sphere, at right $\angle^s$ to a radius drawn from the eye at E; then will its representation *a n b*, on that plane, be a circle.

For through *r*, the centre of the circle A*m*B, draw the plane F*m*G parallel to LM, and join E*r*, *rm*; the former of which will be the axis of the cone of rays flowing to E, and the latter the common section of the two planes AmB, FmG.

Then, because the $\angle Eba$ is measured by half the sum of the arcs EH, KB, or half EKB, it is equal to the $\angle EAB$, which is also measured by half EKB.

Also, because FG is parallel to ab, the $\angle EGF$ is equal to Eba, or EAB; and consequently, by similar $\triangle$s, $Ar : rG :: Fr : rB$, or $Fr \times rG = Ar \times rB$.

But Ar, rB, rm being radii of the same circle AmB, the rectangle $Fr \times rG$ will be $= rm^2$; whence FmG is a circle; as is also the section anb, which is parallel to it (Lem.).

Or the same thing may be shown independently of the Lemma.

For FG being parallel to ab, we shall have $Er : Es :: rF : sa$, and $Er : Es :: rG : sb$; whence, compoundedly, $Er^2 : Es^2 :: Fr \times rG$ (or rm^2) $: sa \times sb$.

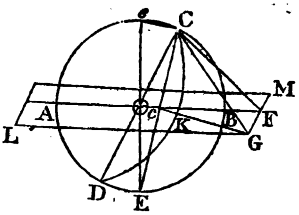
But the plane FmG being parallel to anb, and the plane nr cutting them, sn will be parallel to rm, and consequently $Er^2 : Es^2 :: rm^2 : sn^2$.

Hence, also, by equality, $rm^2 : sa \times sb :: rm^2 : sn^2$; and, therefore, $sa \times sb$ being $= sn^2$, the section anb is a circle. Q. E. D.

Cor. The centres and poles of all circles of the sphere, parallel to the plane of projection, will fall in the centre of the primitive.

THEOREM II.

134. The angle formed by two great circles on the surface of the sphere, is equal to the angle formed by their representatives on the plane of projection.



Let E be the projecting point, or place of the eye, LM the plane of projection, and CBD, CKD two great circles of the sphere, meeting each other in C; then will the projected ∠ be equal to the spherical ∠ BCK.

For, if to the arcs CB, CK there be drawn the tangents CF, CG, meeting the plane LM in F and G, the former of these CF will be projected into *c*F, and the latter CG into *c*G.

And because the $\triangle^s$ EC*e*, EO*c* are right $\angle^d$ at C and O, and have the ∠*e*EC common, the remaining ∠E*e*C will be = the ∠O*c*E, or its opposite ∠C*c*F.

But the ∠ECF, or *c*CF, formed by the chord CE and the tangent CF, being = the ∠E*e*C in the alternate segment, the ∠*c*CF will be = the ∠C*c*F, and the side CF to *c*F.

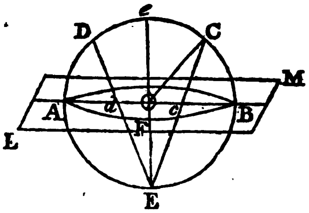
In like manner, it may also be shown, that CG is = *c*G; whence the two sides CF, CG of the △CGF being = to the two sides *c*F, *c*G of the △*c*GF, and the base GF common, the ∠FCG will be = ∠F*c*G.

But since the ∠ made by the intersection of any two arcs is equal to the ∠ made by the tangents of those arcs, drawn from the point of section, the projected ∠ of which *c*F, *c*G are the tangents, is equal to the spherical ∠ BCK.

Cor. The tangent and secant, of any arc of a great circle of the sphere, are represented, on the plane of projection, by right lines equal in length to the former.

THEOREM III.

135. The distance of any projected point of the sphere, from the centre of the primitive, is equal to the semitangent of the arc intercepted between the original point and the pole opposite to the eye.



Let A F B be the primitive circle, lying in the plane of projection L M, E the place of the eye, and C any point on the sphere; then will O *c*, the distance of the projected point *c* from the centre O, be = the semitangent of *e* C.

For, having joined O C, the ∠ *e* E C, or O E *c* at the circumference of the circle *e* A E B, is half the ∠ *e* O C at the centre.

And since the latter of these ∠s *e* O C, is measured by the arc *e* C, the former O E C, will be measured by half that arc.

But O *c* is the tangent of the ∠ O E *c*, to the radius of the sphere E O; whence it is also the tangent of half the ∠ *e* O C, or the semitangent of *e* C.

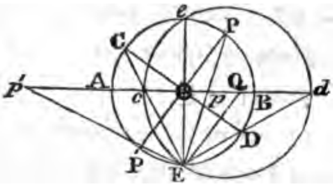
And if any other point D be taken on the opposite side of the pole *e*, it may be shown, in like manner, that O *d* is the semitangent of *e* D. Q. E. D.

Cor. 1. Any arc *e* C of a right circle, commencing at the pole opposite to the eye, is projected into O *c*, the semitangent of that arc.

2. As the poles and extremities of the diameter of any great or small circle are points on the surface of the sphere, their projected distances from the centre of the primitive will be the semitangents of their greatest and least original distances from the pole opposite to the eye.

THEOREM IV.

136. The distances of the projected poles of any oblique great circle from the centre of the primitive, are equal to the tangent and cotangent of half the angle which the two circles make on the sphere; and the distance of their centres is equal to the tangent of the whole of that angle.



Let E be the place of the eye, A *d* the plane of projection, and CD the diameter of a great circle, of which *p*, *p'* are its projected poles, and Q, the middle of the projected diameter *cd*, its centre; then will O*p*, O*p'* be = the tangent and cotangent of ½ the ∠ which this circle and the primitive A B make on the sphere, and OQ = the tangent of the whole of that angle.

For, P, *e* being the poles of CD, AB, the arcs PC, *e*A are quadrants; and, consequently, if *e*C, which is common, be taken away, the remainder *e*P will be = AC.

But O *p* is the tangent of the ∠ O E *p*, or of ½ the arc *e* P, to radius E O; whence it is also the tangent of ½ A C, which is the measure of ½ the inclination of the planes of the two circles A B, C D, or of ½ the ∠ which they make on the sphere.

In like manner, because the ∠ P′E P or *p*′E *p* is a right ∠, the ∠ *p*′ E O will be the complement of the ∠ O E P; and consequently O *p*′ is the cotangent of O E P or ½ *e* O P to radius E O, or of ½ the ∠ which the two circles make on the sphere.

Again, because the lines Q E, Q *d* are equal, being radii of the same circle *e c* E *d*, the outward ∠ O Q E, of the △ Q E *d*, will be double the inward opposite ∠ Q *d* E.

Also, since the ∠^{s} *c* E *d*, *c* O E of the △^{s} E *c d*, E O *c* are right ∠^{s}, and the ∠ E *c* O is common, the remaining ∠ *c* E O will be = Q *d* E.

And because an ∠ at the centre of a circle is double that at the circumference, the ∠ C O *e* is double the ∠ *c* E O, or its equal Q *d* E.

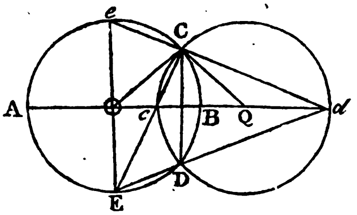
Hence, the ∠^{s} O Q E, C O *e*, being each double the ∠ Q *d* E, are equal; and consequently the ∠^{s} O E Q, C O A, which are their complements, are also equal.

But the ∠ C O A, being the inclination of the planes of the two circles C D, A B, is the measure of the ∠ which they make on the sphere; hence O Q, which is the tangent of the ∠ O E Q, to the radius E O, is also the tangent of the ∠ formed by those circles. Q. E. D.

Cor. It is also evident, from the figure, that the radius E Q, Q C, or Q *d* is equal to the secant of the angle which the two circles make on the sphere.

THEOREM V.

137. The projected radius of any small circle of the sphere, perpendicular to the primitive, is equal to the tangent of its distance from its nearest pole; and the line joining the centres of the two circles is the secant of that distance.



Let E be the place of the eye, A *d* the plane of projection, and C D the diameter of a small circle, perpendicular to the primitive A B; then if its projected diameter *c d* be bisected in Q, the radius Q C will be = the tangent of the arc B C, and O Q will be its secant.

For, since the $\triangle^{s}$ E C *e*, *d* O E are right $\angle^{d}$ at C and O, and have the ∠ *e* common, the remaining ∠ *e* E C, or O E C will be = ∠ *e d* O or C *d* Q.

But Q C being = Q *d*, and O C to O E, the ∠ C *d* Q is = ∠ *d* C Q, and ∠ OEC to ∠ OCE; whence, also, *d* C Q is = ∠ O C E or O C *c*.

And if to each of these equals there be added the ∠ *c* C Q, the whole ∠ O C Q will be = the whole ∠ *c* C *d*.

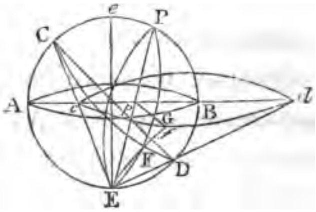
But the ∠ *c* C *d* being a right ∠, the ∠ O C Q is also a right ∠; and consequently Q C is the tangent of B C, and O Q its secant. Q. E. D.

COR. 1. The distance O *c* of the circumference of the projected circle from the centre of the primitive, is = the semitangent of the complement of the arc B C.

2. It also appears from theorem 3, that the radius of any projected great or small circle is $= \frac{1}{2}$ the sum or $\frac{1}{2}$ the difference of the semitangents of its least and greatest distances from the pole opposite to the eye, according as this point is within or without the given circle.

THEOREM VI.

138. Any projected arc of a great circle of the sphere is measured by that arc of the primitive which is cut off by right lines drawn from the projected pole through the two extremities of the given arc.



Let A G B be the primitive circle, lying in the projecting plane A *d*, E the place of the eye, and *c f d* the projection of the great circle C F D; then if right lines *p d*, *p f* be drawn through its pole *p*, the arc *f d* will be measured by G B.

For, since *p d* lies in both the planes A G B, A P B E, it is their common section; and, consequently, will pass through the point B.

In like manner, because *p* G or *p f* is the common section of the planes A G B, P G E, it will pass through the point G.

Hence the points F, D being projected into *f*, *d*, it is plain that the arc F D will be projected into the similar arc *f d*.

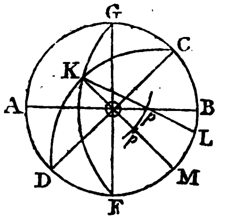
But since PD, EB, PF and EG are each quadrants, if BD, GF, which are common, be taken away, the remainder, or side PB will be = ED, and PG to EF.

And because the opposite $\angle^{s}$ BPG, DEF, which are included by those sides, are equal, the base BG will be = DF.

Whence, FD having been shown to be similar to, or the measure of *fd*, its equal BG will, also, be the measure of *fd*. Q. E. D.

THEOREM VII.

139. Any projected spherical angle, formed by the representatives of two great circles of the sphere, is measured by that arc of the primitive which is cut off by right lines drawn from the angular point through the projected poles of those circles (*b*).



Let GKC be any projected ∠, and *p*, *p'* the poles of the arcs KG, KC by which it is formed; then, if the lines K*p*L, K*p'*M be drawn from the angular point K, to meet the primitive GAFB, the intercepted arc LM will be the measure of the ∠ GKC.

(*b*) For a brief, but neat, treatise on the Stereographic Projection of the Sphere, see an article by Delambre in the Mémoires de l'Institut National, tom. v. also Mémoires de l'Académie de Berlin, for 1779, where this subject is treated analytically with great clearness and elegance.

For, since the angular point of the original arcs, of which K G, K C are the projections, is common to each of them, and 90° distant from their poles, it will be the pole of a great circle of the sphere which passes through the two former poles.

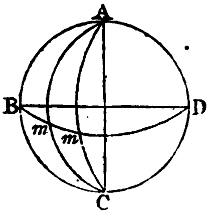
And because the original ∠ on the sphere is measured by that arc of the abovementioned great circle which lies between these two poles, the projected ∠ G K C, which is = the spherical one, will also be measured by the same, or an equivalent arc.

But p, p' being the projection of this arc, and K its projected pole, it is plain, from the last proposition, that the arc L M of the primitive, which is cut off by the lines K p L, K p' M will be the measure of the ∠ G K C. Q. E. D.

MISCELLANEOUS PROBLEMS AND THEOREMS.

PROBLEM I.

140. As four right angles, or 360°, is to the angle B A C, formed by two great circles of the sphere, or to its measure B *m*, so is the surface of the sphere to the lunar area A B C A.



For, let the circle B *m* D, of which A is the pole, be divided into any number of equal parts B *m*, *m m*, &c. and draw the circles A *m* C, A *m* C, &c.

Then will these circles divide the surface of the sphere into the same number of equal parts which the circle B *m* D is divided into.

And since the bases B *m*, *m m*, &c. of the $\triangle^{s}$ B A *m*, *m* A *m*, &c. as well as the sides A B, A *m*, A *m*, &c. are equal, the $\triangle^{s}$ themselves, or their doubles, the lunulæ A B C A, A *m* C A, &c. will be equal.

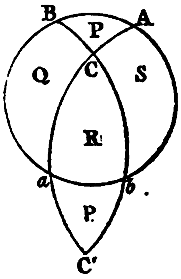
Hence, the sum of the parts B *m*, *m m*, &c. being equal to the whole circumference of the circle B *m* D B, the sum of the lunulæ A B C A, A *m* C A, &c. will also be equal to the whole surface of the sphere.

And, consequently, the circumference of the circle B *m* D B, or 360° : the part, or arc, B*m* : : surface of the sphere : the area of the lune A B C A. Q. E. D.

Cor. Since the surface of the sphere is equal to four times the area of one of its great circles, if d be put $=$ diameter, $c =$ circumference, and $a =$ length of the arc B *m*, by the last mentioned proposition, the area of the lune A B C A will be $= a\,d$, and the area of the whole surface of the sphere $= c\,d$.

PROBLEM II.

The area, or surface, of any spherical triangle, A B C, is equal to the difference between the sum of its three angles and two right angles, multiplied by the radius of the sphere.



For having completed the great circle $ABabA$, produce the sides AC, BC till they meet in the other hemisphere at C'.

Then, because $C'bC$ is equal to bCB, and $C'aC$ to aCA (being each semicircles), if Cb, Ca, which are common, be taken away, the remainders, or sides $C'b$, $C'a$ will be equal to CB, CA.

And since the opposite $\angle^s\ C, C'$ of the lune $CaC'bC$, which are included by those sides, are also equal, the triangle ABC will be equal to the triangle abC'; or P equal to p.

Hence, by the last proposition,

$180^\circ : \angle A :: \frac{1}{2}$ surface of the sphere : $P + Q$

$180^\circ : \angle B :: \frac{1}{2}$ surface of the sphere : $P + S$

$180^\circ : \angle C :: \frac{1}{2}$ surface of the sphere : $P + R$ $(p + R)$

Or, by composition,

$180^\circ : \angle A + \angle B + \angle C :: \frac{1}{2}$ surface of the sphere : $3P + Q + R + S$.

And, by division,

$180^\circ : \angle A + \angle B + \angle C - 180^\circ :: \frac{1}{2}$ surface of the sphere : $3P + Q + R + S - \frac{1}{2}$ surface, or $2P$

Hence, by multiplying the means and extremes of this proportion, there will arise

$180^\circ \times 2P = \frac{1}{2}$ surface of the sphere $\times (\angle A + \angle B + \angle C - 180^\circ)$

Or, putting $180^\circ = \frac{1}{2}c$, and the surface of the sphere $= cd$, as in the Cor. to the last proposition, we shall have P, or

Area $\triangle ABC = \frac{d}{2}(\angle A + \angle B + \angle C - 180^\circ) = r(\angle A + \angle B + \angle C - 180^\circ)$ (c).

(c) This curious theorem, which has been of late years much used, in various geodatical operations, for correcting the angles

COR. Area of the △ in square degrees $= R^\circ (A^\circ + B^\circ + C^\circ - 180^\circ)$ where $R^\circ = 57^\circ.2957795$ the degrees in an arc of equal length with the radius.

Or, if the excess of the three angles of any spherical triangle above two right angles, be required, it may be obtained by the following practical rule:

From the logarithm of the area of the triangle, taken as a plane one, in feet, subtract the constant logarithm 9.3267737, and the remainder will be the logarithm of the excess of the 3 angles of the △ above 180° in seconds, nearly (*d*).

SCHOLIUM. If a, b, c be made to denote the 3 sides of any spherical △ ABC; A, B, C their opposite angles, and π the semicircumference of a great circle of the sphere, of which the radius is r, we shall have, by one of the analogies of Napier,

$$\text{Tan}\ \tfrac{1}{2}(A + B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} \cot \tfrac{1}{2} C$$

From which the following formula may be easily deduced.

$$\text{Cot}\ \tfrac{1}{2}(A + B + C - \pi) = \frac{\cot\frac{1}{2} a \cot\frac{1}{2} b + r \cos C}{\sin C}$$

Which may serve for determining the area of the △,

---

of observation, taken on the surface of the earth, is commonly ascribed to Albert Girard, a Flemish Analyst of the sixth century, who appears to have first given it in nearly the same form as above, in his work entitled *Invention Nouvelle en Algèbre*, published in 1629; where he has also treated of the measures of solid angles. See Montucla Hist. Math. vol. ii. p. 8.

(*d*) For an investigation of this rule, see Trigonometrical Survey of England and Wales, vol. i.

or the spherical excess, when two sides and their included angle are known. And by following the mode of substitution used by Legendre (Elém. de Géom. note 10.) this expression will become

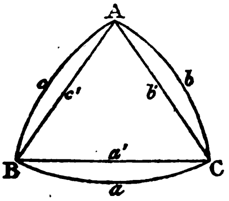
$$\text{Tan}\,\tfrac{1}{4}(\text{A}+\text{B}+\text{C}-\pi) = \frac{1}{r}\sqrt{\tan\frac{a+b+c}{4}\tan\frac{a+b-c}{4}\tan\frac{a+c-b}{4}\tan\frac{b+c-a}{4}}$$

Which elegant theorem was first given by Lhuillier, of Geneva. Legendre, p. 418 of the above work, has also given the following rule for reducing spherical triangles, whose sides are small with respect to the radius of the sphere, to such as are rectilineal.

Any small spherical triangle, whose sides are a, b, c, and their opposite angles A, B, C, always answers to a rectilinear triangle, of which the sides are equal in length to the former, and whose opposite angles are $\text{A}-\frac{1}{3}\varepsilon$, $\text{B}-\frac{1}{3}\varepsilon$ and $\text{C}-\frac{1}{3}\varepsilon$, ε being the excess of the sum of the 3 angles of the spherical triangle above 2 right angles.

PROBLEM III.

142. Two sides and the included angle, of any spherical triangle A B C, being given, to find the angle included by the chords of those sides.



By spherical trigonometry, $\cos a = \sin b \sin c \cos$ sph. $\angle \text{A} + \cos c \cos b$; or $1 - 2\sin^2 \frac{1}{2}a = \sin b \sin c$

$\cos \text{sph.} \angle \text{A} + (1 - 2 \sin^2 \tfrac{1}{2} c) \times (1 - 2 \sin^2 \tfrac{1}{2} b)$; and consequently $2 \sin^2 \tfrac{1}{2} c + 2 \sin^2 \tfrac{1}{2} b - 2 \sin^2 \tfrac{1}{2} a = \sin c \sin b \cos \text{sph.} \angle \text{A} + 4 \sin^2 \tfrac{1}{2} c \sin^2 \tfrac{1}{2} b$.

Again, by plane trigonometry, $a'^2 = b'^2 + c'^2 - 2 b' c' \cos \text{rect.} \angle \text{A}$; and because $a' = 2 \sin \tfrac{1}{2} a$, $b' = 2 \sin \tfrac{1}{2} b$, and $c' = 2 \sin \tfrac{1}{2} c$, we shall have, by substitution, $4 \sin^2 \tfrac{1}{2} a = 4 \sin^2 \tfrac{1}{2} c + 4 \sin^2 \tfrac{1}{2} b - 8 \sin \tfrac{1}{2} c \sin \tfrac{1}{2} b \cos \text{rect.} \angle \text{A}$.

And this value, being substituted in the former equation, will give, $\sin c \sin b \cos \text{sph.} \angle \text{A} + 4 \sin^2 \tfrac{1}{2} c \sin^2 \tfrac{1}{2} b = 4 \sin \tfrac{1}{2} c \sin \tfrac{1}{2} b \cos \text{rect.} \angle \text{A}$.

From which may be readily obtained the following simple expression for the angle contained by the chords, viz.

$\text{Cos rect}^1. \angle \text{A} = \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c + \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c \cos \text{sph.} \angle \text{A}$.

Or, by restoring the value of the radius r,

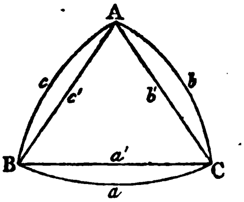
$\text{Cos rect}^1. \angle \text{A} = \frac{1}{r^2} (r \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c + \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c \cos \text{sph.} \angle \text{A})$.

Cor. From the formula here given, it appears that a right or obtuse spherical angle is always greater than the corresponding rectilineal angle.

And that an acute spherical angle is less than its corresponding rectilineal angle, when its cosine is greater than $\frac{r \sin \frac{1}{2} b \sin \frac{1}{2} c}{r^2 - \cos \frac{1}{2} b \cos \frac{1}{2} c}$.

PROBLEM IV.

143. Given the three sides of any spherical triangle A B C, to find the angles contained by the chords of those sides.



By plane trigonometry, cos rect^{l}. ∠ A $= r\left(\frac{b'^2+c'^2-a'^2}{2b'c'}\right)$; but $a' = 2 \sin \frac{1}{2} a$, $b' = 2 \sin \frac{1}{2} b$, and $c' = 2 \sin \frac{1}{2} c$.

Or, $a' = \sqrt{2r(r - \cos a)}$, $b' = \sqrt{2r(r - \cos b)}$, and $c' = \sqrt{2r(r - \cos c)}$; whence, by substitution and reduction, we shall readily obtain

Cos rect. ∠ A $= \frac{1 + \cos a - \cos b - \cos c}{4 \sin \frac{1}{2} b \sin \frac{1}{2} c}$, which, by restoring the value of radius, becomes

Cos rect. ∠ A $= r^2 \frac{r + \cos a - \cos b - \cos c}{4 \sin \frac{1}{2} b \sin \frac{1}{2} c}$. Also cos sph. ∠ A $= \frac{r^2 \cos a - r \cos b \cos c}{\sin b \sin c}$.

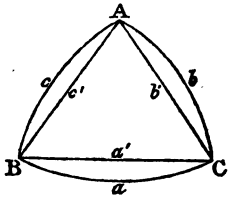
Cor. 1. When the three sides of the spherical triangle are equal, their chords are also equal; in which case, cos rect. ∠ A $= r^2 \frac{r - \cos a}{4 \sin^2 \frac{1}{2} a} = \frac{r}{2} = \cos 60°$, as it ought.

2. Also, since, in the same case, cos sph. ∠ A $= \frac{r \tan \frac{1}{2} a}{\tan a} = \frac{r^2 - \tan^2 \frac{1}{2} a}{2r}$, if $a = 60°$, either of the ∠^{s} will be that of which the cosine is $\frac{1}{3}$ or .3333, &c. that is 70° 31′ 43″; and as the angles are all equal, and their sum greater than 180°, the spherical ∠ A must be greater than $\frac{180°}{3}$, or 60°, which is the rectilineal ∠ A.

PROBLEM V.

144. Given the three angles of any spherical triangle

A B C, to find the angles contained by the chords of the sides.



The sine of ½ any arc being equal to ½ the chord of the whole arc, we shall have, by spherical trigonometry, $a' = 2r\sqrt{\frac{-\cos\frac{1}{2}s\cos(\frac{1}{2}s-A)}{\sin B\sin C}}$, $b' = 2r\sqrt{\frac{-\cos\frac{1}{2}s\cos(\frac{1}{2}s-B)}{\sin A\cdot\sin C}}$, $c' = 2r\sqrt{\frac{-\cos\frac{1}{2}s\cos(\frac{1}{2}s-C)}{\sin A\sin B}}$, where A, B, C denote the three ∠ˢ of the spherical △, and s their sum.

Also, by plane trigonometry, cos rect^l. ∠ A $= r\frac{b'^2 + c'^2 - a'^2}{2b'c'}$; whence, if the former values of a', b', c', be substituted in this equation, we shall obtain, after proper reductions, the following expressions:

Cos rect^l. ∠ A =

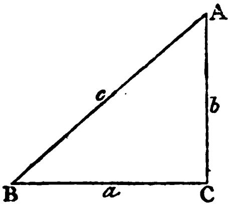
$$\frac{r}{2}\left\{\frac{\sin B\cos(\frac{1}{2}s-B)+\sin C\cos(\frac{1}{2}s-C)-\sin A\cos(\frac{1}{2}s-A)}{\sqrt{\sin B\sin C\cos(\frac{1}{2}s-B)\cos(\frac{1}{2}s-C)}}\right\}$$

which, when A, B, C are equal, becomes cos rect^l. ∠ A $= \frac{r}{2}$, as it ought.

And by observing the order of the letters, the other two angles may be each expressed by a similar form (*e*).

(*e*) The angles formed by the chords, in all the other cases of spherics, might be easily obtained in a similar way; but, as they are less useful, and more complicated in their forms, it was thought proper to omit them.

145. SOLUTIONS OF ALL THE CASES OF RIGHT-ANGLED PLANE TRIANGLES, INDEPENDENTLY OF ANY TABLES.



1. Given the hypothenuse and either of the oblique angles, to find either of the legs.

From the expressions above given, we might, also, readily pass to the solutions of the inverse cases of these problems.

Thus, in prob. III, art. 142, since $\cos\frac{1}{2}b\cos\frac{1}{2}c = \sqrt{r^2-\sin^2\frac{1}{2}b} \times \sqrt{r^2-\sin^2\frac{1}{2}c}$; also $\sin\frac{1}{2}b \times \sin\frac{1}{2}c = \frac{1}{2}b' \times \frac{1}{2}c' = \frac{1}{4}b'c'$, if these values be substituted in the equation cos rect^l. ∠ A = $\sin\frac{1}{2}b\sin\frac{1}{2}c + \cos\frac{1}{2}b\cos\frac{1}{2}c$ cos sph. ∠ A, the radius being considered as 1, we shall have, cos sph. ∠ A = $\frac{\text{cos rect.} \angle \text{A} - \frac{1}{4}b'c'}{\sqrt{(1-\frac{1}{4}b'2)\times(1-\frac{1}{4}c'^2)}}$; in the calculation of which form, the sides b' c' must be so taken that they may be the chords of a circle, having 1 for its radius: which may be easily done, by dividing their numerical values in feet, yards, &c. by such a power of 10, that neither of them shall exceed 2, which is the greatest chord in the circle.

If the △ be isosceles, and of consequence $b = c$, and $b' = c'$, these two formulæ may be reduced to the following: sin $\frac{1}{2}$ rect^l. ∠ A = $\cos\frac{1}{2}b \sin\frac{1}{2}$sph. ∠ A, and $\sin\frac{1}{2}$sph. ∠ A = $\frac{\sin\frac{1}{2}\text{rect.} \angle \text{A}}{\sqrt{1-\frac{1}{4}b'^2}}$.

And if sph. ∠ A be 90°, the same forms will give cos rect^l. ∠ A = $\frac{1}{4}b'c'$. From the first of which expressions it appears, that the vertical ∠ of any isosceles spherical △ is always greater than its corresponding rectilineal angle.

$$a = \frac{c \sin A}{r} = \frac{c \cos B}{r}$$

$$a = c \begin{cases} \dfrac{A^\circ}{R^\circ} - \dfrac{1}{2.3}\left(\dfrac{A^\circ}{R^\circ}\right)^3 + \dfrac{1}{2.3.4.5}\left(\dfrac{A^\circ}{R^\circ}\right)^5 - \dfrac{1}{2.3.4.5.6.7}\left(\dfrac{A^\circ}{R^\circ}\right)^7 + \&c. \\ \text{Or,} \\ 1 - \frac{1}{2}\left(\dfrac{B^\circ}{R^\circ}\right)^2 + \dfrac{1}{2.3.4}\left(\dfrac{B^\circ}{R^\circ}\right)^4 - \dfrac{1}{2.3.4.5.6}\left(\dfrac{B^\circ}{R^\circ}\right)^6 + \&c. \end{cases}$$

And if B° be put in the place of A° in the 1st of these series, and A° in the place of B° in the 2d, they will express the value of b.

Note. A° or B° denote the number of degrees and decimal parts in the length of the arc which measures the ∠ A or B; and $R^\circ = 57^\circ.2957795$ is the number of degrees, &c. in an arc equal in length to the radius. Also the length of any arc A, in parts of the radius, $= r\left(\frac{A^\circ}{R^\circ}\right) = r\left(\frac{A'}{R'}\right) = r\left(\frac{A''}{R''}\right)$.

II. Given the hypothenuse and either of the legs, to find either of the oblique angles.

$$\sin A = \frac{ra}{c},\ \sin B = \frac{rb}{c}$$

$$A^\circ = R^\circ \left\{ \frac{a}{c} + \frac{1}{2.3}\left(\frac{a}{c}\right)^3 + \frac{3}{2.4.5}\left(\frac{a}{c}\right)^5 + \frac{3.5}{2.4.6.7}\left(\frac{a}{c}\right)^7 + \&c. \right\}$$

$$B^\circ = R^\circ \left\{ \frac{b}{c} + \frac{1}{2.3}\left(\frac{b}{c}\right)^3 + \frac{3}{2.4.5}\left(\frac{b}{c}\right)^5 + \frac{3.5}{2.4.6.7}\left(\frac{b}{c}\right)^7 + \&c. \right\}$$

III. Given the hypothenuse and either of the legs, to find the other leg.

$$a = \sqrt{c^2 - b^2},\ b = \sqrt{c^2 + a^2}$$

$$a =$$

$$c - b\left\{ \frac{1}{2}\left(\frac{b}{c}\right) + \frac{1}{2.4}\left(\frac{b}{c}\right)^3 + \frac{3}{2.4.6}\left(\frac{b}{c}\right)^5 + \frac{3.5}{2.4.6.8}\left(\frac{b}{c}\right)^7 \&c. \right\}$$

$$b =$$

$$c - a\left\{\frac{1}{2}\left(\frac{a}{c}\right) + \frac{1}{2.4}\left(\frac{a}{c}\right)^3 + \frac{3}{2.4.6}\left(\frac{a}{c}\right)^5 + \frac{3.5}{2.4.6.8}\left(\frac{a}{c}\right)^7 \text{ \&c.}\right\}$$

IV. Given either of the legs and either of the oblique angles, to find the hypothenuse.

$$c = \frac{a}{r} \sec \text{B} = \frac{b}{r} \sec \text{A}$$

$$c = a\left\{1 + \frac{1}{2}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^2 + \frac{5}{24}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^4 + \frac{61}{720}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^6 + \frac{277}{8064}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^8 + \text{\&c.}\right\}$$

$$c = b\left\{1 + \frac{1}{2}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^2 + \frac{5}{24}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^4 + \frac{61}{720}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^6 + \frac{277}{8064}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^8 + \text{\&c.}\right\}$$

V. Given either of the legs and either of the oblique angles, to find the other leg.

$$a = \frac{b \tan \text{A}}{r} = \frac{b \cot \text{B}}{r}$$

$$a = \begin{cases} b\left\{\frac{\text{A}^\circ}{\text{R}^\circ} + \frac{1}{3}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^3 + \frac{2}{15}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^5 + \frac{17}{315}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^7 + \frac{62}{2835}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^9 \text{ \&c.}\right. \\ \text{Or,} \\ b\left\{\frac{\text{R}^\circ}{\text{B}^\circ} - \frac{1}{3}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right) - \frac{1}{45}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^3 - \frac{2}{945}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^5 - \frac{1}{4725}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^7 \text{ \&c.}\right. \end{cases}$$

And if a be put in the place of b, B° in the place of A° in the first of these series, and A° in the place of B° in the second, they will express the values of b.

VI. Given the two legs, to find either of the oblique angles.

$$\text{Tan A} = \frac{r\,a}{b};\ \cot \text{A} = \frac{r\,b}{a}$$

$$\text{A}^\circ = \begin{cases} \text{R}^\circ\left\{\frac{a}{b} - \frac{1}{3}\left(\frac{a}{b}\right)^3 + \frac{1}{5}\left(\frac{a}{b}\right)^5 - \frac{1}{7}\left(\frac{a}{b}\right)^7 + \frac{1}{9}\left(\frac{a}{b}\right)^9 - \text{\&c.}\right\} \\ \text{Or, } 90^\circ - \\ \text{R}^\circ\left\{\frac{b}{a} - \frac{1}{3}\left(\frac{b}{a}\right)^3 + \frac{1}{5}\left(\frac{b}{a}\right)^5 - \frac{1}{7}\left(\frac{b}{a}\right)^7 + \frac{1}{9}\left(\frac{b}{a}\right)^9 - \text{\&c.}\right\} \end{cases}$$

And if *b* be put in the place of *a*, and *a* in that of *b*, these series will express the value of B°.

VII. Given the two legs, to find the hypothenuse.

$$c = \sqrt{a^2+b^2} = a\sqrt{\left(1+\frac{b^2}{a^2}\right)} = b\sqrt{\left(1+\frac{a^2}{b^2}\right)}$$

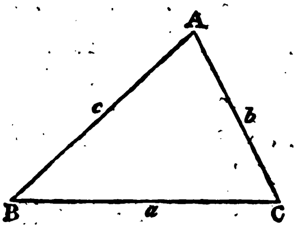
$$c = \begin{cases} a+b\left\{\frac{1}{2}\left(\frac{b}{a}\right)-\frac{1}{2.4}\left(\frac{b}{a}\right)^3+\frac{3}{2.4.6}\left(\frac{b}{a}\right)^5-\frac{3.5}{2.4.6.8}\left(\frac{b}{a}\right)^7\&c.\right\} \\ \text{Or,} \\ b+a\left\{\frac{1}{2}\left(\frac{a}{b}\right)-\frac{1}{2.4}\left(\frac{a}{b}\right)^3+\frac{3}{2.4.6}\left(\frac{a}{b}\right)^5-\frac{3.5}{2.4.6.8}\left(\frac{a}{b}\right)^7\&c.\right\} \end{cases}$$

Note. The following formula, which belongs equally to both the tables, may here also be subjoined, as it will be found useful upon particular occasions:

$$\frac{a+b+c}{r} + \frac{1}{2.3}\left(\frac{a^3+b^3+c^3}{r^3}\right) + \frac{3}{2.4.5}\left(\frac{a^5+b^5+c^5}{r^5}\right) + \frac{3,5}{2.4.6.7}\left(\frac{a^7+b^7+c^7}{r^7}\right)\ \&c. = 3.14159265358979,\ \&c.$$

= the circumference of a circle whose diameter is 1, *a*, *b*, *c* being the halves of the 3 sides of any triangle, and *r* the radius of the circumscribing circle.

146. SOLUTIONS OF ALL THE CASES OF OBLIQUE-ANGLED PLANE TRIANGLES, INDEPENDENTLY OF ANY TABLES.



I. Given a side and two angles, to find either of the other sides.

$$b = a\left\{\frac{\sin \text{B}}{\sin \text{A}} = a\left\{\frac{\cos\,(90^\circ \sim \text{B})}{\cos\,(90^\circ \sim \text{A})}\right.\right.$$

$$b = a\begin{cases} \dfrac{\frac{\text{B}^\circ}{\text{R}^\circ} - \frac{1}{2.3}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^3 + \frac{1}{2.3.4.5}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^5 - \frac{1}{2.3.4.5.6.7}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^7 \&\text{c.}}{\frac{\text{A}^\circ}{\text{R}^\circ} - \frac{1}{2.3}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^3 + \frac{1}{2.3.4.5}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^5 - \frac{1}{2.3.4.5.6.7}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^7 \&\text{c.}} \\ \text{Or,} \\ \dfrac{1 - \frac{1}{2}\left(\frac{90^\circ \sim \text{B}^\circ}{\text{R}^\circ}\right)^2 + \frac{1}{2.3.4}\left(\frac{90^\circ \sim \text{B}^\circ}{\text{R}^\circ}\right)^4 - \frac{1}{2.3.4.5.6}\left(\frac{90^\circ \sim \text{B}^\circ}{\text{R}^\circ}\right)^6}{1 - \frac{1}{2}\left(\frac{90^\circ \sim \text{A}^\circ}{\text{R}^\circ}\right)^2 + \frac{1}{2.3.4}\left(\frac{90^\circ \sim \text{A}^\circ}{\text{R}^\circ}\right)^4 - \frac{1}{2.3.4.5.6}\left(\frac{90^\circ \sim \text{A}^\circ}{\text{R}^\circ}\right)^6} \end{cases}$$

In which case, the numerator of either of these series may be placed over the denominator of the other, when such a combination is found to render them more convergent.

The side a or c may also be expressed in the same manner, using the angles which are similarly situated with respect to these sides, instead of those in the above formula; and the same may be observed in all the other cases when only one side or angle is exhibited.

II. Given two sides and an angle opposite to one of them, to find the remaining side.

$$c = \frac{b}{r}\cos \text{A} \pm \frac{1}{r}\sqrt{r^2 a^2 - b^2 \sin^2 \text{A}}$$

$$c = \begin{cases} b+a-b\left\{\dfrac{b+a}{2\,a}\left(\dfrac{\text{A}^\circ}{\text{R}^\circ}\right)^2 + \dfrac{3\,(b^3+a^3)-4\,a^2\,(b+a)}{2.3.4\,a^3}\left(\dfrac{\text{A}^\circ}{\text{R}^\circ}\right)^4 \&\text{c.}\right\} \\ \text{Or,} \\ b-a-b\left\{\dfrac{b-a}{2\,a}\left(\dfrac{\text{A}^\circ}{\text{R}^\circ}\right)^2 + \dfrac{3\,(b^3-a^3)-4\,a^2\,(b-a)}{2.3.4\,a^3}\left(\dfrac{\text{A}^\circ}{\text{R}^\circ}\right)^4 \&\text{c.}\right\} \end{cases}$$

The first of which series only takes place when a is equal to, or greater than, b; but if a be less than b, either of them will hold, as the question, in this case, admits of two answers.

III. Given two sides and an angle opposite to one of them, to find either of the other angles.

$$\text{Sin A} = \frac{a}{b}\sin \text{B}$$

$$\text{A}^\circ = \frac{a}{b}\text{R}^\circ\left\{\frac{\text{B}^\circ}{\text{R}^\circ} - \frac{b^2-a^2}{2.3\,b^2}\left(\frac{\text{B}^\circ}{\text{R}^\circ}\right)^3 + \frac{(b^2-a^2)^2-8a^2(b^2-a^2)}{2.3.4.5\,b^4}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^5 \text{\&c.}\right\}$$

$$\text{Sin C} = \frac{\sin \text{A}}{a}\left(\frac{b}{r}\cos \text{A} \pm \frac{1}{r}\sqrt{r^2a^2 - b^2\sin^2 \text{A}}\right)$$

$$\text{C}^\circ =$$

$$\left\{\begin{array}{l} \frac{b+a}{a}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right) - \frac{3b^2+4ab+a^2}{2.3\,a^2}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^3 + \frac{30a^2b^2-16a^3b+a^4-15b^4}{2.3.4.5\,a^4} \\ \quad\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^5 \text{\&c.} \\ \qquad\qquad\qquad\qquad \text{Or,} \\ \frac{b-a}{a}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right) + \frac{3b^2-4ab+a^2}{2.3\,a^2}\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^3 + \frac{15b^4+16a^3b-30a^2b^2-a^4}{2.3.4.5\,a^4} \\ \quad\left(\frac{\text{A}^\circ}{\text{R}^\circ}\right)^5 \text{\&c.} \end{array}\right.$$

The first of which series, for C°, only takes place when *a* is equal to, or greater than, *b*; but if *a* be less than *b*, either of them will hold, as the question, in this case, admits of two answers.

IV. Given two sides and the included angle, to find the remaining side.

$$c = \sqrt{(a+b)^2 - \frac{4ab}{r^2}\cos^2 \text{C}}$$

$$c = \sqrt{(a-b)^2 - ab\left\{\left(\frac{\text{C}^\circ}{\text{R}^\circ}\right)^2 - \frac{1}{3.4}\left(\frac{\text{C}^\circ}{\text{R}^\circ}\right)^4 + \frac{1}{3.4.5.6}\left(\frac{\text{C}^\circ}{\text{R}^\circ}\right)^6 \text{\&c.}\right\}}$$

$$c = \sqrt{a^2 + b^2 - \frac{2ab}{r}\cos^2 \text{C}}$$

$$c = (a+b) \times$$

$$\left\{1 - \frac{ab}{2(a+b)^2}\left(\frac{180^\circ - \text{C}^\circ}{\text{R}^\circ}\right)^2 - \frac{3a^2b^2+ab(a+b)^2}{2.3.4\,(a+b)^4}\left(\frac{180^\circ - \text{C}^\circ}{\text{R}^\circ}\right)^4 \text{\&c.}\right\}$$

The first of which series must be used when c° is less than 90°, and the latter when it is greater than 90°.

V. Given two sides and the included angle, to find either of the other angles.

$$\text{Tan } \tfrac{1}{2}\,(\text{A} \sim \text{B}) = \frac{a \sim b}{a+b} \cot \tfrac{1}{2}\,\text{C}$$

$$\text{A}^\circ = 90^\circ - \tfrac{1}{2}\,\text{C}^\circ \pm \text{R}^\circ \times \frac{a \sim b}{a+b}\left\{\frac{2\,\text{R}^\circ}{\text{C}^\circ} - \frac{\text{C}^\circ}{2.3\,\text{R}^\circ} - \frac{1}{8.45}\left(\frac{\text{C}^\circ}{\text{R}^\circ}\right)^3 \&\text{c.}\right\} - \frac{\overline{a \sim b}|^3}{3.\overline{a+b}|^3}\left\{\frac{2\,\text{R}^\circ}{\text{C}^\circ} - \frac{\text{C}^\circ}{2.8\,\text{R}^\circ} - \frac{1}{8.45}\left(\frac{\text{C}^\circ}{\text{R}^\circ}\right)^3 \&\text{c.}\right\}$$

$$+\frac{\overline{a \sim b}|^5}{5.\overline{a+b}|^5}\left\{\frac{2\,\text{R}^\circ}{\text{C}^\circ} - \frac{\text{C}^\circ}{2.3\,\text{R}^\circ} - \frac{1}{8.45}\left(\frac{\text{C}^\circ}{\text{R}^\circ}\right)^3 - \&\text{c.}\right\}$$

Or,

$$\text{A}^\circ = 90^\circ - \tfrac{1}{2}\,\text{C}^\circ \pm \frac{a \sim b}{a+b}\,\text{R}^\circ\left\{\frac{90^\circ - \frac{1}{2}\text{C}^\circ}{\text{R}^\circ} - \frac{2\,(a^2+b^2)}{3\,(a+b)^2}\left(\frac{90^\circ - \frac{1}{2}\text{C}^\circ}{\text{R}^\circ}\right)^3 + \frac{11\,(a^4+b^4)+4\,a^2b^2}{3.5\,(a+b)^4}\left(\frac{90^\circ - \frac{1}{2}\text{C}^\circ}{\text{R}^\circ}\right)^5 \&\text{c.}\right\}$$

In each of which series the upper sign + must be taken when *a* is greater than *b*, and the lower sign — when *a* is less than *b*; observing, in either case, to take the series which is found to be the most convergent.

VI. Given the three sides, to find either of the angles.

$$\text{Cos C} = r\,\frac{a^2+b^2-c^2}{2\,ab}$$

$$\text{C}^\circ = \text{R}^\circ\left\{1 - \tfrac{1}{2}\left(\frac{a^2+b^2-c^2}{2\,ab}\right)^2 + \frac{1}{2.8.4}\left(\frac{a^2+b^2-c^2}{2\,ab}\right)^4 - \frac{1}{2.3.4.5.6}\left(\frac{a^2+b^2-c^2}{2\,ab}\right)^6 \&\text{c.}\right\}$$

Which series will always converge, whatever may be the values of the sides *a*, *b*, *c*.

Note. If the angle C, in the 4th or 5th case, be very obtuse, or near 180°, and, of consequence, the remaining angles A and B small, the side c, and either of these angles, may be found, to a considerable degree of exactness, by means of the following formulæ:

$$c = a + b - \frac{a\,b}{2\,(a+b)}\left(\frac{180^\circ - \text{C}^\circ}{\text{R}^\circ}\right)^2$$

$$\text{A}^\circ = \frac{a\,(180^\circ - \text{C}^\circ)}{a+b}\left\{1 + \frac{b\,(a-b)}{6\,(a+b)^2}\left(\frac{180^\circ - \text{C}^\circ}{\text{R}^\circ}\right)^2\right\}$$

The angle and side, in this case, may also be expressed, in series, by means of the formula given in page 337, as follows:

$$\text{A} = \frac{a}{b}\sin\text{C} + \frac{a^2}{2\,b^2}\sin 2\,\text{C} + \frac{a^3}{3\,b^3}\sin 3\,\text{C} + \frac{a^4}{4\,b^4}\sin 4\,\text{C}\ \&\text{c.}$$

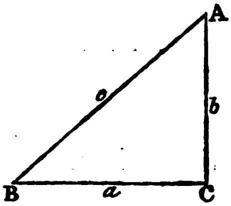
Or,

$$\text{B} = \frac{b}{a}\sin\text{C} + \frac{b^2}{2\,a^2}\sin 2\,\text{C} + \frac{b^3}{3\,a^3}\sin 3\,\text{C} + \frac{b^4}{4\,a^4}\sin 4\,\text{C}\ \&\text{c.}$$

$$\text{Log}\ c = \log a -$$

$$\frac{1}{\text{M}}\left\{\frac{b}{a}\cos\text{C} + \frac{b^3}{2\,a^2}\cos 2\,\text{C} + \frac{b^3}{3\,a^3}\cos 3\,\text{C} + \&\text{c.}\right.$$

147. To the tables last given, there may likewise be added the following solutions of some particular cases of right-angled plane triangles, when one part is given, and the sum or difference of two others.



1. $c + a = \frac{b^2}{c-a}$; $c - a = \frac{b^2}{c+a}$

2. $\text{Tan}\,\frac{1}{2}\,\text{A} = \frac{(c-b)\,r}{a} = \frac{r\,a}{c+b}$

3. $\text{Sin } \frac{1}{2} \text{A} = r \sqrt{\frac{c-b}{2c}}$; $\cos \frac{1}{2} \text{A} = r \sqrt{\frac{c+b}{2c}}$

4. $c + a = \frac{c-a}{r^2} \cot^2 \frac{1}{2} \text{B}$; $c - a = \frac{c+a}{r^2} \tan^2 \frac{1}{2} \text{B}$

5. $a + b = \frac{a-b}{r} \cot (45° - \text{B}) = \frac{a-b}{r} \tan (45° + \text{B})$

6. $a - b = \frac{a+b}{r} \tan (45° - \text{B}) = \frac{a+b}{r} \cot (45° + \text{B})$

7. $\text{Sin } (45° + \text{B}) = \frac{(a+b)\, r}{c \sqrt{2}}$; $\sin (45° - \text{B}) = \frac{(a-b)\, r}{c \sqrt{2}}$

8. $c - a = b \tan \frac{1}{2} \text{B}$, $c - b = a \tan \frac{1}{2} \text{A}$.

148. The following additional formulæ for right-angled plane triangles may also be subjoined to those given above.

1. $\text{Sin } (\text{B} - \text{A}) = \frac{r\,(b+a) \times (b-a)}{c^2}$; $\cos (\text{B} - \text{A}) = \frac{2r\,a\,b}{c^2}$

2. $\text{Tan } (\text{B} - \text{A}) = \frac{r\,(b+a) \times (b-a)}{2\,a\,b}$; $\tan (\text{A} - \text{B}) = \frac{r\,(a^2 - b^2)}{2\,a\,b}$

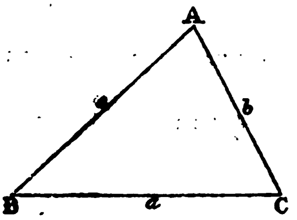
3. $\text{Sin } 2\,\text{A} = \sin 2\,\text{B} = \frac{2r\,a\,b}{c^2} = \frac{2r\,a\,b}{a^2 + b^2} = \frac{2\,a\,b}{(a+b)^2 - 2\,a\,b}$

4. $\text{Cos } 2\,\text{A} = \frac{r\,(b^2 - a^2)}{c^2} = \frac{r\,(b^2 - a^2)}{a^2 + b^2}$; $\cos 2\,\text{B} = \frac{r\,(a^2 - b^2)}{c^2}$

5. $\text{Sin } 4\,\text{A} = \frac{4\,r\,a\,b\,(b^2 - a^2)}{c^4}$; $\sin 4\,\text{A} = \frac{4\,r\,a\,b\,(a^2 - b^2)}{c^4}$.

From which latter formula, it appears, that 4 A is greater than 180°, when a is greater than b.

149. To these may also be subjoined the following formulæ, which serve for the solutions of some particular cases of oblique-angled plane triangles, when the sum or differences of certain parts are given, instead of the parts themselves.



1. $\text{Sin}\ \frac{1}{2}(\text{B} \sim \text{C}) = \frac{b \sim c}{a} \cos \frac{1}{2}\text{A}$

2. $\text{Cos}\ \frac{1}{2}(\text{B} \sim \text{C}) = \frac{b + c}{a} \sin \frac{1}{2}\text{A}$

3. $\text{Cot}\ \frac{1}{2}\text{A} = \frac{(a+b)+c}{(a+b)-c} \tan \frac{1}{2}\text{B}$

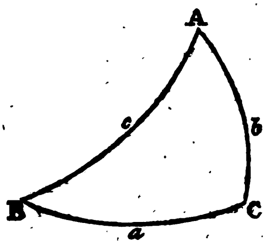
4. $\text{Tan}\ \frac{1}{2}\text{A} = \frac{c+(a-b)}{c-(a-b)} \tan \frac{1}{2}\text{B}$

5. $\text{Tan}\ \frac{1}{2}\text{A} = \frac{b \sim c}{b + c} \cot \frac{1}{2}(\text{B} \sim \text{C})$

6. $b \sim c = \frac{b+c}{r^2} \tan \frac{1}{2}\text{A} \tan \frac{1}{2}(\text{B} \sim \text{C})$

7. $b + c = \frac{b \sim c}{r^2} \cot \frac{1}{2}\text{A} \cot \frac{1}{2}(\text{B} \sim \text{C})$

150. Similar formulæ may likewise be given for the solutions of some particular cases of right-angled spherical triangles; the most useful of which are the following:



1. $r^2\ \text{Tan}\ \frac{1}{2}(c - a) = \tan^2 \frac{1}{2} b \cot \frac{1}{2}(c + a)$

Or, $r^2\ \text{Tan}\ \frac{1}{2}(c + a) = \tan^2 \frac{1}{2} b \cot \frac{1}{2}(c - a)$

2. $r^2\ \text{Sin}\ (c - a) = \sin b \cos a \tan \frac{1}{2}\text{B}$

Or, $r^2\ \text{Sin}\ (c - a) = \tan b \cos c \tan \frac{1}{2}\text{B}$

3. $r\ \text{Cot}\ \frac{1}{2}(45^\circ + \frac{1}{2}\text{A}) = \tan \frac{1}{2} b \cot \frac{1}{2}(c + a)$

4. $\text{Cos}\ (a \sim b) = 2 \cos c - \cos (a + b)$

5. $r\ \text{Tan}\ (45^\circ + \frac{1}{2}\text{A}) = \tan \frac{1}{2} b \cot \frac{1}{2}(c - a)$

6. $r^2\ \text{Sin}\ (c - a) = \tan^2 \frac{1}{2}\text{B} \sin (c + a)$

7. $r^2\ \text{Sin}\ (c + a) = \cot^2 \frac{1}{2}\text{B} \sin (c - a)$

8. $r^2 \text{Cos}\,(A \sim B) = -\cos\,(A+B)\cot^2 \tfrac{1}{2}c$

9. $r^2 \text{Cos}\,(A+B) = -\cos\,(A \sim B)\tan^2 \tfrac{1}{2}c$

10. $r^2 \text{Tan}\,\tfrac{1}{2}(c-a) = \cot^2 \tfrac{1}{2}(45^\circ + \tfrac{1}{2}A)\tan \tfrac{1}{2}(c+a)$

11. $r\,\text{Tan}\,\tfrac{1}{2}b = \cot\,(45^\circ + \tfrac{1}{2}A)\tan \tfrac{1}{2}(c+a)$

12. $r^2 \text{Tan}\,\tfrac{1}{2}(c+a) = \tan^2 (45^\circ + \tfrac{1}{2}A)\tan \tfrac{1}{2}(c-a)$

13. $r\,\text{Tan}\,\tfrac{1}{2}b = \tan\,(45^\circ + \tfrac{1}{2}A)\tan \tfrac{1}{2}(c-a)$

14. $r^2 \text{Tan}\,\tfrac{1}{2}(c+a) = \tan^2 \tfrac{1}{2}b \cot \tfrac{1}{2}(c-a)$

15. $\text{Cos}\;C = \tfrac{1}{2}\cos\,(a+b) + \tfrac{1}{2}\cos\,(a \sim b)$

16. $\text{Sin}\;b = \tfrac{1}{2}\cos\,(c \sim B) - \tfrac{1}{2}\cos\,(c+B)$

17. $\text{Sin}\;a = \tfrac{1}{2}\cos\,(c \sim A) - \tfrac{1}{2}\cos\,(c+A)$

18. $\text{Cos}\;A = \tfrac{1}{2}\sin\,(a+B) - \tfrac{1}{2}\sin\,(a-B)$

19. $\text{Cos}\;B = \tfrac{1}{2}\sin\,(b+A) - \tfrac{1}{2}\sin\,(b-A)$

Among which formulæ it is evident, that the five last, being formed by addition and subtraction only, may be calculated by means of the natural sines and cosines.

To these we may likewise add the four following forms, which are applicable to any spherical triangle whatever.

$$\frac{\text{Sin}\,\frac{1}{2}(b-c)}{\sin\frac{1}{2}a} = \frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}; \quad \frac{\text{Sin}\,\frac{1}{2}(b+c)}{\sin\frac{1}{2}a} = \frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}$$

$$\frac{\text{Cos}\,\frac{1}{2}(b-c)}{\cos\frac{1}{2}a} = \frac{\sin\frac{1}{2}(B+C)}{\cos\frac{1}{2}a}; \quad \frac{\text{Cos}\,\frac{1}{2}(b+c)}{\cos\frac{1}{2}a} = \frac{\cos\frac{1}{2}(B+C)}{\sin\frac{1}{2}A}$$

151. OF THE INCREMENTS AND FLUXIONS OF THE SINES AND TANGENTS OF ARCS, OR ANGLES.

As formulæ of this kind are frequently employed in astronomy and the higher branches of mathematics, in order to show the changes that take place in the sides and angles of triangles, from small variations of some of their parts, I shall here subjoin such of them as will be found sufficient to answer most of the purposes to which they are usually applied.

Thus, if z' be made to represent the increment of the arc z, and $\sin' z$ the increment of its sine, &c. we shall readily obtain the following expressions; which are only the formulæ in article 26 under another form.

1. $r \sin' z = 2 \sin \frac{1}{2} z' \cos (z + \frac{1}{2} z')$
2. $-r \cos' z = 2 \sin \frac{1}{2} z' \sin (z + \frac{1}{2} z')$
3. $\text{Tan}' z = \frac{r^2 \sin z'}{\cos z \cos (z + z')}$
4. $-\text{Cot}' z = \frac{r^2 \sin z'}{\sin z \sin (z + z')}$
5. $\text{Sin}'^2 z = \sin z' \sin (2z + z')$
6. $-\text{Cos}'^2 z = \sin z' \sin (2z + z')$
7. $\text{Tan}'^2 z = \frac{\sin z' \sin (2z + z')}{\cos^2 z \cos^2 (z + z')}$
8. $-\text{Cot}'^2 z = \frac{\sin z' \sin (2z + z')}{\sin^2 z \sin^2 (z + z')}$

Which expressions are rigorously exact, whatever may be the magnitude of the quantity z'; and if the second member of either of these equations is to be employed with a negative sign, we must substitute $(z - z')$ instead of $(z + z')$, $(z - \frac{1}{2} z')$ for $(z + \frac{1}{2} z')$, and $(2z - z')$ for $(2z + z')$.

152. It is also evident, that if z', in these expressions, be taken indefinitely small, or within less than any assignable limit of 0, its sine may be considered as equal to the arc, and its cosine to the radius: whence, by proper substitutions, we shall readily obtain the following formulæ, for the fluxions of the sine, tangent, &c. of any arc or angle; being the same as those usually given by the writers on this subject.

1. $\dot{\text{Sin}}\, z = \dot{z} \cos z = \dot{z} \sqrt{r^2 - \sin^2 z}$

2. $- \dot{\text{Cos}}\, z = \dot{z} \sin z = \dot{z} \sqrt{r^2 - \cos^2 z}$

3. $\dot{\text{Tan}}\, z = \frac{\dot{z}}{\cos^2 z} = \frac{\dot{z}}{r^2 - \sin^2 z} = \frac{\dot{z} \sec^2 z}{r^2}$

4. $- \dot{\text{Cot}}\, z = \frac{\dot{z}}{\sin^2 z} = \frac{\dot{z}}{r^2 - \cos^2 z} = \frac{\dot{z} \operatorname{cosec}^2 z}{r^2}$

5. $\dot{\text{Sin}}^2\, z = 2 \dot{z} \sin z \cos z = \dot{z} \sin 2 z$

6. $- \dot{\text{Cos}}^2\, z = 2 \dot{z} \sin z \cos z = \dot{z} \sin 2 z$

7. $\dot{\text{Tan}}^2 z = 2 \tan z \,\dot{\tan}\, z = 2 \tan z \times \frac{\dot{z}}{\cos^2 z}$

8. $- \dot{\text{Cot}}^2\, z = 2 \cot z \,\dot{\cot}\, z = 2 \cot z \times \frac{\dot{z}}{\sin^2 z}$

Which formulæ are only to be employed when the variation of the arc is extremely small, as the result, when its increase or decrease is considerable, is often very erroneous (*f*).

153. From these latter expressions we may likewise readily obtain the following, in which the fluxion of the arc is expressed in terms of its sine, tangent, &c.

1. $\dot{z} = \frac{r \,\dot{\sin}\, z}{\sqrt{r^2 - \sin^2 z}}$

2. $\dot{z} = - \frac{r \,\dot{\cos}\, z}{\sqrt{r^2 - \cos^2 z}}$

3. $\dot{z} = \frac{r^2 \,\dot{\tan}\, z}{r^2 + \tan^2 z}$

4. $\dot{z} = - \frac{r^2 \,\dot{\cot}\, z}{r^2 + \cot^2 z}$

(*f*) For a more copious collection of formulæ of this kind, both incremental and fluxional, with their applications, see *Traité de Trigonométrie de Cagnoli*, 2d edition, where this subject is very ably and fully discussed.

$$5.\ \dot{z} = \frac{r^2 \sec \dot{z}}{\sec z \sqrt{\sec^2 z - r^2}}.$$

$$6.\ \dot{z} = \frac{r \operatorname{vers} \dot{z}}{\sqrt{2 r \operatorname{vers} z - \operatorname{vers}^2 z}}.$$

And if these formulæ are expanded, and the fluents of each term be taken, they will give the usual series for the arc, in terms of its sine, tangent, &c.

154. Some of the trigonometrical formulæ, given in the former articles of this work, may also be applied to the solution of the various cases of quadratic and cubic equations, as follows:

SOLUTIONS OF QUADRATIC EQUATIONS.

$$1.\ x^2 + p x - q = 0.$$

$$\text{Put } \frac{2}{p} \sqrt{q} = \tan z.$$

$$\text{Then } x = \begin{cases} + \sqrt{q} \times \tan \frac{1}{2} z, \\ \text{or,} \\ - \sqrt{q} \times \cot \frac{1}{2} z. \end{cases}$$

Or, putting

$$10 + \log 2 + \tfrac{1}{2} \log q - \log p = \log \tan z.$$

$$\text{Then } \log x = \begin{cases} + \frac{1}{2} \log q + \log \tan \frac{1}{2} z - 10, \\ \text{or,} \\ - \frac{1}{2} \log q - \log \cot \frac{1}{2} z - 10. \end{cases}$$

$$2.\ x^2 - p x - q = 0.$$

$$\text{Put } \frac{2}{p} \sqrt{q} = \tan z,$$

$$\text{Then } x = \begin{cases} + \sqrt{q} \times \cot \frac{1}{2} z, \\ \text{or,} \\ - \sqrt{q} \times \tan \frac{1}{2} z. \end{cases}$$

Or, putting

$$10 + \log 2 + \tfrac{1}{2} \log q - \log p = \log \tan z.$$

$$\text{Then } \log x = \begin{cases} +\tfrac{1}{2}\log q + \log\cot\tfrac{1}{2}z - 10, \\ \qquad\qquad \text{or,} \\ -\tfrac{1}{2}\log q - \log\tan\tfrac{1}{2}z - 10. \end{cases}$$

3. $x^2 + px + q = 0.$

Put $\frac{2}{p}\sqrt{q} = \sin z.$

$$\text{Then } x = \begin{cases} -\sqrt{q}\times\tan\tfrac{1}{2}z, \\ \qquad \text{or,} \\ -\sqrt{q}\times\cot\tfrac{1}{2}z. \end{cases}$$

Or, putting

$$10 + \log 2 + \tfrac{1}{2}\log q - \log p = \log\sin z,$$

$$\text{Then } \log x = \begin{cases} -\tfrac{1}{2}\log q - \log\tan\tfrac{1}{2}z - 10, \\ \qquad\qquad \text{or,} \\ -\tfrac{1}{2}\log q - \log\cot\tfrac{1}{2}z - 10. \end{cases}$$

4. $x^2 - px + q = 0.$

Put $\frac{2}{p}\sqrt{q} = \sin z.$

$$\text{Then } x = \begin{cases} +\sqrt{q}\times\tan\tfrac{1}{2}z, \\ \qquad \text{or,} \\ +\sqrt{q}\times\cot\tfrac{1}{2}z. \end{cases}$$

Or, putting

$$10 + \log 2 + \tfrac{1}{2}\log q - \log p = \log\sin z.$$

$$\text{Then } \log x = \begin{cases} +\tfrac{1}{2}\log q + \log\tan\tfrac{1}{2}z - 10, \\ \qquad\qquad \text{or,} \\ +\tfrac{1}{2}\log q + \log\cot\tfrac{1}{2}z - 10. \end{cases}$$

In the two latter of which cases, if $\frac{2}{p}\sqrt{q}$ be greater than 1, or $10 + \log 2 + \frac{1}{2}\log q - \log p$ comes out greater than 10, the two roots, or values of x, will be impossible.

155. SOLUTIONS OF CUBIC EQUATIONS.

$$1.\ x^3 + px - q = 0.$$

$$\text{Put } \frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \tan z, \text{ and } \sqrt[3]{\tan(45^\circ - \tfrac{1}{2}z)} = \tan u;$$

$$\text{Then } x = 2\sqrt{\frac{p}{3}} \times \cot 2u.$$

Or, putting

$$\text{Log } \frac{q}{2} + 10 - \frac{3}{2}\log\frac{p}{2} = \log\tan z,$$

And

$$\frac{1}{3}(\log\tan\overline{45^\circ - \tfrac{1}{2}z} + 20) = \log\tan u,$$

$$\text{Then } \log x = \tfrac{1}{2}\log\frac{4p}{3} + \log\cot 2u - 10.$$

$$2.\ x^3 + px + q = 0.$$

$$\text{Put } \frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \tan z, \text{ and } \sqrt[3]{\tan(45^\circ - \tfrac{1}{2}z)} = \tan u,$$

$$\text{Then } x = -2\sqrt{\frac{p}{3}} \times \cot 2u.$$

Or, putting

$$\text{Log } \frac{q}{2} + 10 - \frac{3}{2}\log\frac{p}{3} = \log\tan z,$$

And

$$\frac{1}{3}(\log\tan\overline{45^\circ - \tfrac{1}{2}z} + 20) = \log\tan u,$$

$$\text{Then } \log x = 10 - \tfrac{1}{2}\log\frac{4p}{3} - \log\cot 2u.$$

$$3.\ x^3 - px - q = 0.$$

This form has 2 cases,* according as $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}}$ is less or greater than 1.

$$\text{In the 1st case, put } \frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}} = \cos z.$$

$$\text{And } \sqrt[3]{\tan 45^\circ - \tfrac{1}{2}z)} = \tan u;$$

$$\text{Then } x = 2\sqrt{\frac{p}{3}} \times \operatorname{cosec} 2u.$$

Or, putting

$$10 + \frac{3}{2}\log\frac{p}{3} - \log\frac{q}{2} = \log\cos z,$$

And

$$\frac{1}{3}\left(\log\tan\overline{45^\circ - \tfrac{1}{2}z} + 20\right) = \log\tan u;$$

$$\text{Then } \log x = 10 + \log\frac{4p}{3} - \log\sin 2u.$$

In the 2d case, put $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \cos z$, and x will have the 3 following values:

$$x = +2\sqrt{\frac{p}{3}} \times \cos\frac{z}{3}$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos\left(60^\circ - \frac{z}{3}\right)$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos\left(60^\circ + \frac{z}{3}\right)$$

Or,

$$\text{Log } x = \tfrac{1}{2}\log\frac{4p}{3} + \log\cos\frac{z}{3} - 10,$$

$$\text{Log } x = \tfrac{1}{2}\log\frac{4p}{3} + \log\cos\left(60^\circ - \frac{z}{3}\right) - 10,$$

$$\text{Log } x = \tfrac{1}{2}\log\frac{4p}{3} + \log\cos\left(60^\circ + \frac{z}{3}\right) - 10,$$

Taking the value of x, answering to log x, positive in the first, and negative in the two latter.

4. $x^3 - px + q = 0$.

This form, like the one above, has also two cases, according as $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}}$ is less or greater than 1.

In the 1st case, put $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}} = \cos z$,

And $\sqrt[3]{\tan\left(45^\circ - \frac{1}{2}z\right)} = \tan u$, as before;

$$\text{Then } x = -2\sqrt{\frac{p}{3}}\ \text{cosec}\ 2u.$$

Or, putting

$$10+\frac{3}{2}\log\frac{p}{3}-\log\frac{q}{2}=\log\cos z,$$

And

$$\frac{1}{3}\left\{\log\left(\tan 45^\circ-\tfrac{1}{2}z\right)+20\right\}=\log\tan u;$$

$$\text{Then, }-\log x=10+\log\frac{4p}{3}-\log\sin 2u.$$

In the 2d case, put $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}}=\cos z$, and x will have the 3 following values:

$$x=-2\sqrt{\frac{p}{3}}\times\cos\frac{z}{3}$$

$$x=+2\sqrt{\frac{p}{3}}\times\cos\left(60^\circ-\frac{z}{3}\right)$$

$$x=+2\sqrt{\frac{p}{3}}\times\cos\left(60^\circ+\frac{z}{3}\right).$$

Or,

$$\text{Log } x=\tfrac{1}{2}\log\frac{4p}{3}+\log\cos\frac{z}{3}-10,$$

$$\text{Log } x=\tfrac{1}{2}\log\frac{4p}{3}+\log\cos\left(60^\circ-\frac{z}{3}\right)-10,$$

$$\text{Log } x=\tfrac{1}{2}\log\frac{4p}{3}+\log\cos\left(60^\circ+\frac{z}{3}\right)-10,$$

Taking the value of x, answering to log x, negative in the first, and positive in the two latter.

As an example of this mode of solution, in what is usually called the *Irreducible Case of Cubic Equations*, let $x^3-3x=1$, to find its 3 roots.

Here $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}}=\frac{1}{2}\left(\frac{3}{3}\right)^{\frac{3}{2}}=\frac{1}{2}=.5=\cos 60^\circ=z,$

$$\text{Hence}\begin{cases}x=2\sqrt{\frac{p}{3}}\times\cos\frac{z}{3}=2\cos 20^\circ=1.8793852\\ x=-2\sqrt{\frac{p}{3}}\times\cos\left(60^\circ-\frac{z}{3}\right)=-2\cos 40^\circ=-1.5320888\\ x=-2\sqrt{\frac{p}{3}}\times\cos\left(60^\circ+\frac{z}{3}\right)=-2\cos 80^\circ=-\ .3472964.\end{cases}$$

Also, let $x^3 - 3x = -1$, to find its three roots.

Here, as before, $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = .5 = \cos 60° = z$,

$$\text{Hence}\left\{\begin{array}{l} x = -2\sqrt{\frac{p}{3}} \times \cos\frac{z}{3} = -2\cos 20° = -1.8793852 \\ x = -2\sqrt{\frac{p}{3}} \times \cos\left(60° - \frac{z}{3}\right) = 2\cos 40° = 1.5320888 \\ x = -2\sqrt{\frac{p}{3}} \times \cos\left(60° + \frac{z}{3}\right) = 2\cos 80° = 0.3472964. \end{array}\right.$$

Where the roots are the negatives of those of the first case (*g*).

OF THE ADMEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

156. Having treated, pretty fully, in the former part of this work, of the methods of measuring elevations and depressions geometrically, I shall here subjoin one of the most easy practical rules for determining the same thing by means of the barometer and thermometer, which is a mode frequently used at present; and though not founded upon such sure and well-established principles as the former, is nevertheless susceptible, when performed by a skilful observer, with good instruments, of a considerable degree of accuracy.

For this purpose, the person who undertakes these observations should be provided with two portable barometers, of the best construction (both filled with mercury of the same specific gravity), on which, by

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(*g*) For the mode of investigating these kinds of formulæ, see Cagnoli Traité de Trig. and article Irreducible Case, given by the author, in the Supplement to Hutton's Mathematical Dictionary.

means of a nonius, properly adapted to the scale, he may read off the heights of the mercurial columns to the 200dth part of an inch. Each of the barometers must, also, have an attached thermometer, set in the wooden frame, in the same manner as the barometer is, and having their balls of nearly the same diameter as that of the barometer tube: besides which, there must be two other thermometers detached from the barometers.

Then, one of these barometers, with its attached and detached thermometers, is to be placed in the shade at the top of the eminence whose height is required, while the other remains below; and when they have continued in their places a sufficient time for the detached thermometers to acquire the temperature of the air, or till the fluid is stationary, the observer on the eminence must note down the height of the mercurial column in the barometer, as well as the temperatures exhibited by the attached and detached thermometers; and, at the same time, another observer must make the like observations on the instruments below.

This being done, the altitude of the object, at the top and bottom of which the instruments were placed, may be ascertained by observing the following precepts, or the practical rule which is deduced from them.

1. The height through which we must rise to produce a fall of mercury in the barometer, is inversely proportional to the density of the air, or to the height of the mercury in the barometer.

2. When the barometer stands at 30 inches, and the

air and quicksilver are of the temperature of 32°, we must rise through 87 feet to produce a depression of $\frac{1}{10}$th of an inch.

3. But if the air be of a different temperature, this 87 feet must be increased, or diminished, by about $\frac{21}{100}$ of a foot for every degree of difference of temperature above or below 32°.

4. Every degree of difference in the temperatures of the mercury at the two stations makes a change of 2.883 feet, or 2 feet 10 inches in the elevation.

Hence the following Rule.

1. Take the difference of the barometric heights, in tenths of an inch, and multiply the result by 30.

2. Multiply the difference between 32° and the mean temperature of the air, by .21, and find the sum or difference of this product and 87 feet, according as the temperature is above or below 32°.

3. Multiply this sum, or difference, by the former product, and the result, divided by the mean of the barometric heights, will give the approximated elevation.

4. Multiply the difference of the mercurial temperatures by 2.833 feet, and add this product to the approximated elevation, if that of the upper barometer be the greatest, or otherwise subtract it, and the result will be the corrected elevation.

Or, the same rule may be expressed algebraically thus:

$$\text{A} = \frac{30\,\text{D}\,(87 \pm 0.21\,d)}{m} \pm 2.833\,\delta$$

Where *d* is the difference between 32° and the mean temperature of the air, D the difference of the barometric heights in tenths of an inch, m the mean barometric height, δ the difference between the mercurial temperatures, and A the corrected altitude.

For example, suppose the mercury in the barometer, at the lower station, was at 29.4 inches, its temperature 50° of Fahrenheit's thermometer, and the temperature of the air 45°: also the height of the mercury, at the upper station, 25.19 inches, its temperature 46°, and the temperature of the air 39°.

$$\text{Then}\begin{cases} d = \dfrac{(45^0 - 32^0) + (39^0 - 32^0)}{2} = 10 \\ \text{D} = (29.4 - 25.19) \times 10 = 42.1 \\ \delta = 50 - 46 \text{ - - - - - - - } = 4. \\ m = \frac{1}{2}\,(29.4 + 25.19) \text{ - - } = 27.295 \end{cases}$$

$$\text{And A} = \frac{30 \times 42.1 \times (87 + .21 \times 10)}{27.295} + 4 \times 2.833$$

$= 4111.91$ feet, or 685.32 fathoms, the correct altitude.

And if two or three sets of observations be made at each station, after short intervals of time, and the mean of the results be taken, the probability of error will be much diminished.

It may here be added, that the method of measuring altitudes by means of the barometer, was first distinctly pointed out by Halley, in No. 181 of the Philosophical Transactions; but it was not till long after his time that the method was turned to any real use. The chief improvements are due to M. de Luc, who published, at Geneva, the result of his experiments and inquiries, made on the high mountains of Switzerland, in a treatise on the barometer and thermometer; and also in the Philosophical Transactions. Other valuable papers on this subject have likewise been given by Maskelyne, Horsley, Sir George Shuckburgh, and General Roy, in the different volumes of these Transactions.

But the most complete account, that has yet appeared, of this useful and interesting part of the subject, may be found in the additions to chap. 15, vol. iii, of the *Traité Elémentaire d'Astronomie Physique*, by Biot, who has there entered into all the details that can be wished for, on this head, both with respect to the demonstrations of the formulæ, and their several applications.

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